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On a new class of parallel sequencing situations and related games

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Abstract: This paper considers a special class of sequencing situations with two parallel machines in which each agent has precisely two jobs to be processed, one on each machine. The costs of an agent depend linearly on the final completion time of his jobs. We describe a procedure that provides an optimal processing order of the jobs. Furthermore, we study cooperative games arising from these sequencing situations. Our main result is balancedness of these games.

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1 Introduction

Sequencing or scheduling situations find their origin in the processing and manufacturing industries, but also arise in computing, business and service industries. The specific problems we consider in this paper are those arising from situations where some complementary jobs, in the sense that they cannot be processed on the same machine, need to be finished in order to obtain the final output.

Typical examples are the production processes of cars, or other manufacturing activities where different elements of the final output are processed on different machines before being assembled. Other examples include scheduling tasks that have to be processed by the C.P.U. of a computer in order to obtain the final result.

To clarify the problem consider a construction firm that has to install some different services like electricity, gas, water, etc. in some houses. The houses can be of a different type and the time needed to provide each service to the houses can also be different. Each service is provided by one specific specialist and hence, the same service cannot be provided to two houses at the same time. The firm incurs costs for each house until the house is ready for sale, i.e., until all services have been provided. These costs will vary between the houses.

The first problem the firm in the above situation faces is the problem of finding an optimal schedule in providing the services to the houses, i.e., a schedule that minimizes the total costs.

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A subsequent problem the firm may face is how to allocate the total costs in the optimal schedule to the houses. The need for such an allocation can be, for example, for accounting reasons. Let us assume that initially there exists an initial schedule in providing the services to the houses. Say, the schedule based on the first-come, first-served principle. Then the optimal schedule can differ from the initial one. In that case, there are houses that in the optimal schedule would be offered for sale later than in the initial schedule due to other, high priority houses (e.g. higher market prices) being earlier. In this case, one may argue that the high priority houses are responsible for part of the increases in the costs for low priority houses. The remaining problem is which allocations of the total costs may be considered "fair".

The "fair" allocation of costs is one of the main issues that is addressed in cooperative game theory. Establishing a relation between scheduling problems and cooperative games will enable us to obtain "fair" cost allocations. By assuming that there exists an initial schedule before the machines (services) start processing (providing the service), one can establish a relation between cooperative games and sequencing situations in the following way. Under the assumption that each job (house)is owned by an agent, a group of agents (a coalition) can save costs by rearranging their jobs in a way that is admissible with respect to this initial schedule. By defining the value of a coalition as the maximal cost savings a coalition can make in this way, we obtain a cooperative sequencing game related to the sequencing situation.

In this regard the core of a cooperative game comes to mind. Roughly speaking, a core allocation divides the costs in such a way that, for each group of agents their total costs in the optimal schedule plus the additional costs allocated to them does not exceed the total costs they can obtain by exchanging their places in the initial schedule in an admissible way. Core allocations, however, need not always exist. If a core allocation exists for a cooperative game, the game is called balanced. In this paper we show that under specific conditions on parallel scheduling problems the corresponding games are balanced.

The above game-theoretic approach to sequencing situations was initiated by *Curiel, Pederzoli and Tijs* (1989) by considering the class of one-machine sequencing situations. It was shown that the corresponding sequencing games are convex and, thus, balanced. *Hamers, Borm and Tijs* (1995) extended the class of one machine sequencing situations considered by *Curiel et al.* (1989) by imposing ready times on the jobs. The corresponding sequencing games are balanced, but are not necessarily convex. Similar results are also obtained in *Borm, Fiestras-Janeiro, Hamers, Sánchez and Voorneveld* (1999) in which due dates are imposed on the jobs. Instead of imposing intrinsic restrictions on the jobs, *Van den Nouweland, Krabbenborg and Potters* (1992) extended the number of machines. Here m-machines sequencing situations are considered associated to flow shops situations with a so-called dominant machine. Convexity was established for a special subclass. In general, however, the corresponding games need not be balanced. Finally, *Hamers, Klijn and Suijs* (1999) consider m-parallel and identical machines sequencing situations, and prove the balancedness of the related games in some special cases.

This paper considers sequencing situations with two parallel machines. Contrary to other papers in this field, it is assumed that each agent owns two jobs to be processed, one on each machine. The costs of an agent depend linearly on the final completion time of his jobs. In other words, it depends on the time an agent has to wait until both his jobs have been processed. A formal description of the model and some results in providing optimal schedules of the jobs of the agents on the machines are presented in Section 2.

In Section 3, we introduce games related to this type of sequencing situations. These games are studied with respect to convexity and σ_0-component additivity (cf. *Curiel, Potters, Rajendra Prasad, Tijs and Veltman* (1993, 1994)).

In Section 4, the games are studied with respect to balancedness. We derive conditions such that the core of the games is non-empty: a specific marginal vector lies in the core. The proof of this result is technically quite involved. Most of these technicalities are concentrated in an Appendix.

Finally, in Section 5, we conclude with some remarks on possible extensions to the general m-parallel machines case.

2 The model

In this section, we start with describing the specific sequencing situations with 2 parallel machines that we study in the paper. After that, we provide some results on the optimal processing order of the jobs on the machines for some special situations.

The set of the two machines is denoted by $M = \{1, 2\}$. There is a finite set of agents $N = \{1, ..., n\}$. We assume that each agent has 2 jobs to be processed, one on each machine.

Moreover, we assume that each machine starts processing at time 0 and by the vector $(p_i, q_i)_{i\in N}$ we denote the processing times of the jobs of every agent i, $p_i \geq 0$ for the job to be processed on machine 1 and $q_i \geq 0$ for the job to be processed on machine 2. We also assume that there is an initial scheme (π, φ) of the jobs on the machines where π and φ are the initial orders for the first and the second machine, respectively. Formally, π and φ are bijections from N to $\{1, 2, ..., n\}$ where $\pi(i) = s$ and $\varphi(i) = t$ mean that initially, player i has a job in position s on machine 1 and a job in position t on machine 2 in the initial queues before the machines. Let $\prod(N)$ be the set of orders of N, i.e., bijections from N to $\{1, 2, ..., n\}$. Then $\prod(N) \times \prod(N)$denotes the set of possible schemes.

Concerning the costs of spending time in the system, every agent has a linear cost function $c_i : [0, \infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t$ where $\alpha_i > 0$,and where t represents the time player i has to wait to have both his jobs processed.

A 2 parallel machines sequencing situation is a 5-tuple $(M, N, (\pi, \varphi), (\alpha_i)_{i\in N}, (p_i, q_i)_{i\in N})$ and we will refer to it as a 2$-$PS situation. Notice that 2$-$PS situations generalize the class of sequencing situations studied by *Curiel et al.* (1989).

Let $(\sigma, \tau) \in \prod(N) \times \prod(N)$ be a scheme. We denote by $C_i(\sigma) := \sum\limits_{j\in N:\sigma(j)\leq\sigma(i)} p_j$ the completion time $C_i(\sigma)$ of the job of agent i on the first machine with respect to the order σ. Similarly, $C_i(\tau) := \sum\limits_{j\in N:\tau(j)\leq\tau(i)} q_i$ denotes the completion time of the job of agent i on the second machine with respect to τ. Considering as relevant for every player the moment he can leave the system, we consider the final completion time with respect to (σ, τ), that is $C_i(\sigma, \tau) := \max\{C_i(\sigma), C_i(\tau)\}$. Then the total costs of the agents with respect to (σ, τ) can be written as

$$c_N(\sigma, \tau) := \sum_{i\in N} \alpha_i C_i(\sigma, \tau).$$

A scheme $(\hat{\sigma}, \hat{\tau}) \in \prod(N) \times \prod(N)$ is called optimal for N if total costs are minimized, i.e.,

$$c_N(\hat{\sigma}, \hat{\tau}) = \min_{(\sigma,\tau) \in \prod(N) \times \prod(N)} c_N(\sigma, \tau).$$

The following proposition can be useful in finding an optimal scheme.

Proposition 2.1 *For a $2-PS$ situation:*
1) There exists an optimal scheme $(\hat{\sigma}, \hat{\tau})$ with $\hat{\sigma} = \hat{\tau}$.
2) If $p_i = q_i$ for all $i \in N$ then for any optimal scheme $(\hat{\sigma}, \hat{\tau})$ it holds that $\hat{\sigma} = \hat{\tau}$

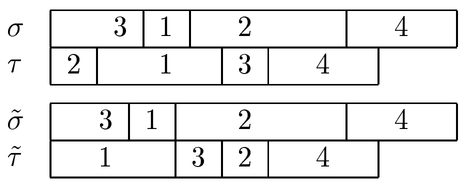
Proof. 1) We are done if we show that for any scheme (σ, τ) with $\sigma \neq \tau$ we can construct a weakly better scheme $(\tilde{\sigma}, \tilde{\tau})$ with $\tilde{\sigma} = \tilde{\tau}$, i.e., $c_N(\sigma, \tau) \geq c_N(\tilde{\sigma}, \tilde{\tau})$.

Let (σ, τ) be a scheme with $\sigma \neq \tau$. Then going from the last position to the first position, we will find a position s with different players on the two machines, i.e.,
$s = \max\{t \in \{1, ..., n\} : \exists\, i, j \in N \; i \neq j$ such that $\sigma(i) = \tau(j) = t\}$.
Related to position s we define the sets of players
$I(s) = \{i \in N : \sigma(i) = s$ or $\tau(i) = s\}$ and
$I^*(s) = \{i^* \in I(s) : C_{i^*}(\sigma, \tau) \geq C_i(\sigma, \tau)$ for all $i \in I(s)\}$.

We construct a weakly better scheme $(\tilde{\sigma}, \tilde{\tau})$ as follows. We choose a player $i^* \in I^*(s)$. Without loss of generality, assume that i^* has position s on the first machine, i.e., $\sigma(i^*) = s$. Then define, $\tilde{\sigma} := \sigma$ and

$$\tilde{\tau}(i) := \begin{cases} \tau(i) - 1 & \text{if} \quad \tau(i) > \tau(i^*) \text{ and } \tau(i) \leq s \\ s & \text{if} \quad i = i^* \\ \tau(i) & \text{if} \quad \tau(i) < \tau(i^*) \text{ or } \tau(i) > s \end{cases}.$$

Hence, $\tilde{\tau}(i^*) = \tilde{\sigma}(i^*) = s$. It is obvious that on the first machine $C_i(\tilde{\sigma}) = C_i(\sigma)$ for all $i \in N$. It is easy to check that on the second machine $C_i(\tilde{\tau}) \leq C_i(\tau)$ for all $i \in N \backslash \{i^*\}$, and $C_{i^*}(\sigma, \tau) \geq C_{i^*}(\tilde{\sigma}, \tilde{\tau})$. Graphically:



where $s = 3$ and $i^* = 2$.

Considering now the scheme $(\tilde{\sigma}, \tilde{\tau})$, by repeating the same argument we obtain a scheme as desired.

2) Let (σ, τ) be a scheme with $\sigma \neq \tau$. We construct a strictly better scheme $(\tilde{\sigma}, \tilde{\tau})$ with $\tilde{\sigma} = \tilde{\tau}$. We use the same argument as in 1). It is easy to see that from $p_i = q_i$ for all $i \in N$ it follows that $I(s) = I^*(s)$. Constructing the same new scheme $(\tilde{\sigma}, \tilde{\tau})$ as before, we realize that $C_{i^*}(\tilde{\sigma}, \tilde{\tau}) = C_{i^*}(\sigma, \tau)$ and for player $j^* \in I^*(s)$, $j^* \neq i^*$ it holds $C_{j^*}(\tilde{\sigma}, \tilde{\tau}) < C_{j^*}(\sigma, \tau)$. Consequently $c_N(\tilde{\sigma}, \tilde{\tau}) < c_N(\sigma, \tau)$. □

Although the problem of finding an optimal scheme for the general case is NP-hard we can present the following result for some specific problems. Following *Smith* (1956) we define $u_i^1 := \frac{\alpha_i}{p_i}$ and $u_i^2 := \frac{\alpha_i}{q_i}$ as the urgency of player i on machine 1 and 2, respectively.

Proposition 2.2 *For a 2−PS situation with $p_i = q_i$ for all $i \in N$ it holds that $(\hat{\sigma}, \hat{\sigma})$ is an optimal scheme if and only if*

$$\hat{\sigma}(i) \le \hat{\sigma}(j) \Leftrightarrow u_i \ge u_j \text{ for all } i, j \in N \tag{1}$$

where $u_i := u_i^1 = u_i^2$ for all $i \in N$.

Proof. By assumption of $p_i = q_i$ for all $i \in N$ we can define $u_i := u_i^1 = u_i^2$ for all $i \in N$. First we show the only if part. Suppose $(\hat{\sigma}, \hat{\sigma})$ is an optimal scheme that does not satisfy (1). Then there exist two players $i, j \in N$ such that $\hat{\sigma}(i) = \hat{\sigma}(j) + 1$ and $u_i > u_j$. Construct the scheme $(\tilde{\sigma}, \tilde{\sigma})$ where players i and j are switched on both machines. In other words,

$$\tilde{\sigma}(k) := \begin{cases} \hat{\sigma}(j) & \text{if} \quad k = i \\ \hat{\sigma}(i) & \text{if} \quad k = j \\ \hat{\sigma}(k) & \text{if} \quad k \neq i, j \end{cases} .$$

Then it holds that

$$\begin{aligned} c_N(\hat{\sigma}, \hat{\sigma}) - c_N(\tilde{\sigma}, \tilde{\sigma}) &= \alpha_j C_j(\hat{\sigma}, \hat{\sigma}) + \alpha_i C_i(\hat{\sigma}, \hat{\sigma}) - \alpha_i C_i(\tilde{\sigma}, \tilde{\sigma}) - \alpha_j C_j(\tilde{\sigma}, \tilde{\sigma}) \\ &= \alpha_i p_j - \alpha_j p_i > 0 \end{aligned}$$

where the last inequality follows from $u_i > u_j$. Hence $(\hat{\sigma}, \hat{\sigma})$ is not optimal.

Next we show the if part. Let $(\hat{\sigma}, \hat{\sigma})$ satisfy (1) and let (σ, τ) be an optimal scheme. Then by proposition 2.1, $\sigma = \tau$. From the only if part (σ, τ) satisfies (1). Now we can obtain $(\hat{\sigma}, \hat{\sigma})$ from (σ, τ) by switching on both machines adjacent pairs i, j with $u_i = u_j$. This will leave the total costs unchanged, which implies that $(\hat{\sigma}, \hat{\sigma})$ is an optimal scheme. □

3 Sequencing games arising from 2−PS situations

In this section we introduce two classes of cooperative games that arise from the sequencing situations discussed in section 2. A transferable utility game (a game, for short) is a pair (N, v) where N is a finite set of players and v is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, and 2^N the collection of all subsets or coalitions of N . If no confusion can arise a game (N, v) will be denoted by its characteristic function v.

Let $(M, N, (\pi, \varphi), (\alpha_i)_{i \in N}, (p_i, q_i)_{i \in N})$ be a 2−PS situation. The maximal cost savings of a set of players $S \subseteq N$ depend on the set of admissible rearrangements of this set of agents S. We call a scheme (σ, τ) to be an admissible rearrangement for S with respect to (π, φ) if it satisfies: two agents $i, j \in S$ can only switch in one machine if all agents in between i and j on that machine with respect to the initial order on that machine are also members of S. Formally, first we define the set of predecessors of player $i \in N$ with respect to an order $\sigma \in \prod(N)$ as $P_i(\sigma) := \{j \in N : \sigma(j) < \sigma(i)\}$. Now, given the initial order $\pi \in \prod(N)$, an admissible order for S on machine 1, is a bijection $\sigma \in \prod(N)$ such that $P_i(\sigma) = P_i(\pi)$ for all $i \in N \backslash S$. Similarly, an admissible order for S on machine 2 is a bijection $\tau \in \prod(N)$ such that $P_i(\tau) = P_i(\varphi)$ for all $i \in N \backslash S$.

Let $\mathcal{A}^1(S)$ and $\mathcal{A}^2(S)$ denote the set of admissible rearrangements on machine 1 and machine 2, respectively. The set $\mathcal{A}^1(S) \times \mathcal{A}^2(S)$ is called the set of admissible schemes for S. In other words, we consider an scheme to be admissible for S if each agent outside S has the same completion time on each machine as in the initial order. Moreover, the agents of S are not allowed to jump over players outside S.

Then, given a $2-$PS situation, the corresponding $2-$*PS game* (N, v) is defined in such a way that the worth of a coalition $S \subseteq N$ is equal to the maximal cost savings the coalition can achieve by means of admissible schemes. Formally,

$$v(S) := \max_{(\sigma,\tau)\in\mathcal{A}^1(S)\times\mathcal{A}^2(S)} \sum_{i\in S} \alpha_i \left(C_i(\pi,\varphi) - C_i(\sigma,\tau)\right).$$

It is straightforward that any $2-$PS game v is *monotonic*, i.e., for all $S \subset T \subseteq N$, it holds that $v(S) \leq v(T)$. This is true since $\mathcal{A}^1(S) \times \mathcal{A}^2(S) \subseteq \mathcal{A}^1(T) \times \mathcal{A}^2(T)$.

It is also easy to see that v is a *superadditive* game, i.e., for all $S, T \subseteq N$, with $S \cap T = \varnothing$, it holds that $v(S) + v(T) \leq v(S \cup T)$. This follows immediately from the observation that for any admissible scheme $(\sigma,\tau) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ for S and any admissible scheme $(\bar{\sigma},\bar{\tau}) \in \mathcal{A}^1(T) \times \mathcal{A}^2(T)$ for T there exists an admissible scheme $(\tilde{\sigma},\tilde{\tau}) \in \mathcal{A}^1(S \cup T) \times \mathcal{A}^2(S \cup T)$ for $S \cup T$ with:
$\tilde{\sigma}(i) := \sigma(i)$ and $\tilde{\tau}(i) := \tau(i)$ for all $i \in S$,
$\tilde{\sigma}(i) := \bar{\sigma}(i)$ and $\tilde{\tau}(i) := \bar{\tau}(i)$ for all $i \in T$.

Curiel et al. (1993) introduced the class of σ_0-component additive games, a well known class of games which incorporates many 1-machine sequencing situations, that is balanced.

Given an order $\sigma_0 \in \prod(N)$, a bijection from N to $\{1, 2, ..., n\}$, a cooperative game v is called σ_0-component additive if the following three conditions are satisfied:

1) $v(i) = 0$ for all $i \in N$,
2) v is superadditive, and
3) $v(S) = \sum_{T\in S/\sigma_0} v(T)$, where S/σ_0 is the set of all maximally connected components of S.

A coalition S is called connected with respect to σ_0 if for all $i, j \in S$ and $k \in N$ such that $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ it holds that $k \in S$.

Proposition 3.1 *Let $(M, N, (\pi,\varphi), (\alpha_i)_{i\in N}, (p_i, q_i)_{i\in N})$ be a $2-$PS situation with $\pi = \varphi$. Then the corresponding game (N, v) is π-component additive.*

Proof. Conditions 1 and 2 hold for any initial scheme (π,φ). We prove condition 3. First, observe that by superadditivity $v(S) \geq \sum_{T\in S/\pi} v(T)$. The reverse inequality follows from.

$$\begin{aligned} v(S) &= \sum_{i\in S} \alpha_i \left(C_i(\pi,\varphi) - C_i(\hat{\sigma},\hat{\tau})\right) \\ &= \sum_{T\in S/\pi} \sum_{i\in T} \alpha_i \left(C_i(\pi,\varphi) - C_i(\hat{\sigma},\hat{\tau})\right) \\ &\leq \sum_{T\in S/\pi} v(T), \end{aligned}$$

where $(\hat{\sigma},\hat{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ is an optimal scheme for coalition S. The first equality follows from the definition of the game (N, v). The second equality follows from the observation that $\pi = \varphi$ and hence, $S/\pi = S/\varphi$. Consequently $(\hat{\sigma},\hat{\tau}) \in \mathcal{A}^1(T) \times \mathcal{A}^2(T)$ for all $T \in S/\pi$, i.e., $(\hat{\sigma},\hat{\tau})$ also determines a scheme for each $T \in S/\pi$. Moreover, the inequality follows from $(\hat{\sigma},\hat{\tau})$ is an optimal scheme for S, but does not need to be optimal for all $T \in S/\pi$. □

The results presented so far are based on the structure of the problem, without the need of any information concerning optimal schemes for a coalition $S \subseteq N$. In general, as has been stated before, the problem of finding the optimal scheme for the grand coalition N is NP-hard. Similarly, the problem of finding the optimal scheme for a coalition $S \subset N$ is NP-hard, too. Hence, the worth of a coalition $S \subseteq N$ cannot be easily described. Therefore, we henceforth restrict our attention to simple 2−PS situations.

Definition 3.1 *A 2−PS situation $(M, N, (\pi, \varphi), (\alpha_i)_{i\in N}, (p_i, q_i)_{i\in N})$ is called simple if*
$p := p_i = q_i = p_j = q_j$ for all $i, j \in N$ and
$\alpha := \alpha_i$ for all $i \in N$.

Without loss of generality, we restrict ourselves to simple 2−PS situations with $\alpha = 1$ and $p = 1$. Then the cost savings associated to any switch between players are either 1 or 0 or −1. Furthermore, for a player $i \in N$ and an order $\sigma \in \prod(N)$ it holds that $C_i(\sigma) = \sigma(i)$. We denote simple 2−PS situations with the 3−tuple $(M, N, (\pi, \varphi))$.

Using propositions 2.1 and 2.2, in simple 2−PS situations, an optimal scheme $(\hat{\sigma}, \hat{\tau}) \in \prod(N) \times \prod(N)$ for N can be derived easily. For every optimal scheme it holds that $\hat{\sigma} = \hat{\tau}$ because of proposition 2.1 and $p_i = q_i$ for all $i \in N$. More precisely, by proposition 2.2 any scheme with $\hat{\sigma} = \hat{\tau}$ will be optimal since for any pair of players $i, j \in N, i \neq j$ it holds that $u_i = u_j$. The rich structure of simple 2−PS situations enables us to provide a simple expression for the worth of the grand coalition N in the corresponding *simple 2−PS game*:

$$\begin{aligned} v(N) &= \max_{(\sigma,\tau)\in\prod(N)\times\prod(N)} \sum_{i\in N} \alpha_i \left(C_i(\pi,\varphi) - C_i(\sigma,\tau)\right) \\ &= \sum_{i\in N} \left(C_i(\pi,\varphi) - C_i(\hat{\sigma},\hat{\tau})\right) \\ &= \sum_{i\in N} \left(C_i(\pi,\varphi) - \max\{\hat{\sigma}(i), \hat{\tau}(i)\}\right) \\ &= \sum_{i\in N} C_i(\pi,\varphi) - (1 + 2 + ... + |S|), \end{aligned}$$

where the last inequality follows easily by the observation that the costs associated to optimal schemes equal $(1 + 2 + ... + |S|)$.

For an arbitrary coalition $S \subset N$, however, it is still difficult to find an optimal scheme $(\hat{\sigma}, \hat{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ or an easy expression for the worth of S. In the next example we give a simple 2−PS situation with an optimal scheme for coalition $S \subset N$ that cannot simply be derived from propositions 2.1 and 2.2.

Example 3.1 Consider the simple 2−PS situation with
$N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the initial scheme given by

π	1	2	3	4	5	6	7	8
φ	7	8	5	6	3	4	1	2

The optimal scheme for $S = \{1, 2, 3, 5, 6, 7, 8\}$ is

$\hat{\sigma}$	3	1	2	×	5	6	7	8
$\hat{\tau}$	3	5	6	7	8	×	1	2

For convenience, in this example and in the following, we leave out all players that are not in the coalition S. By definition of admissible scheme they will be in the same position as in the initial scheme. △

In general, 2−PS games arising from 2−PS situations need not be σ_0-component additive, not even in simple 2−PS situations as the following example shows.

Example 3.2 Consider the simple 2−PS situation with
$N = \{1, 2, 3, 4, 5\}$, and the initial scheme given by

π	1	2	3	4	5
φ	4	1	5	3	2

Suppose that the simple 2−PS game is σ_0-component additive.

1) Note that $v(14) = v(34) = v(35) = 1$ and $v(i) = 0$ for all $i \in N$. So $\{14\}$, $\{34\}$ and $\{35\}$ have to be connected with respect to σ_0.

2) Note also that $v(234) = 2$, $v(34) = 1$, $v(23) = v(24) = 0$ and $v(i) = 0$ for all $i \in N$. So $\{234\}$ has to be connected with respect to σ_0.

So, from 1) and 2) it follows that $\{14\}$, $\{34\}$, $\{35\}$ and $\{234\}$ have to be connected with respect to σ_0, but there is no ordering of the five players that makes that possible. Hence, v is not σ_0-component additive. $\triangle$

A TU-game v is convex if $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and all $S \subset T \subset N \backslash \{i\}$. As the following example shows 2−PS games neither need to be convex, not even in simple 2−PS situations.

Example 3.3 Consider the simple 2−PS situation with:

$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and the initial scheme given by:

π	5	6	7	1	2	3	8	9	4
φ	7	8	3	4	5	6	1	9	2

Take $S = \{1, 3\}$, $T = \{1, 3, 4, 5, 6\}$ and $i = 2$.

The optimal schemes are:

$\hat{\sigma}$	×	×	×	3	1	2	×	×	×	for $S \cup \{i\}$,
$\hat{\tau}$	×	×	3	×	×	×	1	×	2	

$\hat{\sigma}$	5	6	×	1	×	3	×	×	4	for T and
$\hat{\tau}$	×	×	5	6	1	3	4	×	×	

$\hat{\sigma}$	5	6	×	3	1	2	×	×	4	for $T \cup \{i\}$.
$\hat{\tau}$	×	×	5	3	6	1	4	×	2	

$v(T \cup \{i\}) - v(T) = 6 - 6 < 2 - 0 = v(S \cup \{i\}) - v(S)$.

Hence, v is not convex. $\triangle$

A game v is said to be balanced if the core $C(v)$ is non empty. The core of the game v consists of the payoff vectors $x \in \mathbb{R}^N$ satisfying the conditions $x(S) \geq v(S)$ for all $S \in 2^N$ and $x(N) = v(N)$.

Although v does not need to be convex nor σ_0-component additive, not even in simple 2−PS situations, as examples 3.2 and 3.3 show, we can give core elements for both examples: $(0, 0, 1, 2, 2)$ and $(0, 0, 4, 4, 0, 0, 2, 4, 1)$, respectively.

Notice that the games (N, v) arising from 1 machine situations are convex and balanced. Moreover, games arising from 2−PS situations with $\pi = \varphi$ are π-component additive games, and as is shown in *Curiel et al.* (1993) the β−rule gives a core element.

For our study of balancedness of 2−PS games, we introduce a new game (N, w) that will represent some kind of optimistic expectations for a coalition $S \subset N$, since in the new game the players can switch freely on each machine, even if they are not connected. So we call the game w arising from a 2−PS situation, the optimistic 2−PS game.

This games will be useful to deal with the problem of finding an optimal scheme for a coalition $S \subset N$, which is NP-hard, even in simple 2−PS situations.

Before defining the game we introduce some notation. For any order $\sigma \in \prod(N)$ we define $\sigma^S \in \prod(S)$ as the bijection from S to $\{1, 2, ..., |S|\}$, where the relative positions of the players with respect to σ remain unchanged, i.e., $\sigma^S(i) := \sigma(i) - |\{P_i(\sigma) \cap N \backslash S\}|$ for all $i \in S$.

For any 5-tuple $(M, N, (\pi, \varphi), (\alpha_i)_{i \in N}, (p_i, q_i)_{i \in N})$ and a coalition $S \subseteq N$, we can associate an induced 2−PS situation to S denoted by the 5-tuple $(M, N, (\pi^S, \varphi^S), (\alpha_i)_{i \in S}, (p_i, q_i)_{i \in S})$. In this induced situation the set of players is S, and the set of admissible rearrangements for S is $\prod(S) \times \prod(S)$. The corresponding 2−PS game associated to this induced 2−PS situation is denoted by (S, v^S). We call the game (S, v^S) the *induced 2−PS game to S*.

The worth of a coalition $S \subseteq N$ in the *optimistic game* (N, w) is now defined to be the worth of the "grand coalition S" in the corresponding induced 2−PS game (S, v^S) to S. Hence $w(S) := v^S(S)$ for all $S \subseteq N$. Clearly $w(N) = v^N(N) = v(N)$.

Obviously, for a simple 2−PS situation $(M, N, (\pi, \varphi))$, any induced 2−PS situation to a coalition S, $(M, N, (\pi^S, \varphi^S))$ is also a simple 2−PS situation. Consequently, the game w enables us to work with useful expressions concerning the worth of a coalition $S \subset N$ in the game w. In fact,

$$\begin{aligned} w(S) = \quad v^S(S) \quad &= \sum_{i \in S} C_i(\pi^S, \varphi^S) - (1 + 2 + ... + |S|) \\ &= \sum_{i \in S} C_i(\pi^S, \varphi^S) - \tfrac{|S|(|S|+1)}{2} \end{aligned} \tag{2}$$

The following example clarifies the difference between v and w.

Example 3.4 Consider the simple 2−PS situation with

$N = \{1, 2, 3, 4, 5, 6\}$, and the initial scheme given by

π	4	1	5	2	6	3
φ	3	2	5	4	6	1

Let $S = \{1, 2, 3\}$. Then for the induced 2−PS situation to S, the initial scheme is given by

π^S	1	2	3
φ^S	3	2	1

Clearly $v(S) = 0$ and $w(S) = v^S(S) = 2$. △

The game w, as the game v, does not need to be σ_0-component additive nor convex, not even in simple 2−PS situations. This is shown in the next two examples.

Example 3.5 Consider the simple 2−PS situation with

$N = \{1, 2, 3\}$, and the initial scheme given by

π	1	2	3
φ	3	2	1

The characteristic function w is:

$$w(S) = \begin{cases} 0 & \text{if} \quad |S| \leq 1 \\ 1 & \text{if} \quad |S| = 2 \\ 2 & \text{if} \quad |S| = 3 \end{cases}$$

The game w is not σ_0-component additive since there is no ordering of the three players for which coalitions $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$ are connected. △

Example 3.6 Consider the simple 2−PS situation with

$N = \{1, 2, 3, 4\}$, and the initial scheme given by

π	1	2	3	4
φ	4	2	3	1

Let $S = \{2,3\}$, $T = \{1,2,3\}$ and $i = 4$.
Then $w(T \cup \{i\}) - w(T) = 3 - 2 < 2 - 0 = w(S \cup \{i\}) - w(S)$.
So, the game w is not convex. $\triangle$

4 On balancedness for simple $2-$PS situations

In this section we present the main result of the paper: the games arising from simple $2-$PS situations $(M, N, (\pi, \varphi))$ are balanced.

The proof runs as follows. First, we show that $w(S) \geq v(S)$ for all $S \subseteq N$. Clearly $w(N) = v(N)$ and hence $C(w) \subseteq C(v)$. Second, we show that a specific marginal vector is in the core of the game (N, w). Hence, since $C(w) \subseteq C(v)$, the same marginal vector is also a core allocation of (N, v).

Theorem 4.1 shows the relation between the a 2-PS game and the related optimistic game.

Theorem 4.1 *Let (N, v) be the $2-$PS game and (N, w) be the optimistic $2-$PS game arising from a simple $2-$PS situation $(M, N, (\pi, \varphi))$, then it holds that $w(N) = v(N)$ and $w(S) \geq v(S)$ for all $S \subset N$.*

Proof. Clearly $w(N) = v(N)$ and $w(\emptyset) = 0$. Let $S \subset N$, $S \neq \emptyset$, $S \neq N$. By lemma 6.1 there exists an optimal scheme $(\hat{\sigma}, \hat{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ for S such that for all $i \in S$ it holds that

if $\pi(i) \geq \varphi(i)$ then $\hat{\sigma}(i) \geq \hat{\tau}(i)$ and
if $\pi(i) \leq \varphi(i)$ then $\hat{\sigma}(i) \leq \hat{\tau}(i)$.

Let $(\hat{\sigma}, \hat{\tau})$ be such a scheme. We define the following sets:

$$S_1 := \left\{ \begin{array}{ll} i \in S: & \pi(i) > \varphi(i) \text{ or} \\ & \pi(i) = \varphi(i) \text{ and } \hat{\sigma}(i) \geq \hat{\tau}(i) \end{array} \right\},$$

$$S_2 := \left\{ \begin{array}{ll} i \in S: & \pi(i) < \varphi(i) \text{ or} \\ & \pi(i) = \varphi(i) \text{ and } \hat{\sigma}(i) < \hat{\tau}(i) \end{array} \right\}.$$

Note that S_1 or S_2 can be the empty set, and they form a partition of S. Then, for any $i \in S_1$ it holds that

$C_i(\pi, \varphi) = |\{j \in N \backslash S : \pi(j) < \pi(i)\}| + \pi^S(i)$ and
$C_i(\hat{\sigma}, \hat{\tau}) = |\{j \in N \backslash S : \pi(j) < \pi(i)\}| + \hat{\sigma}^S(i)$.

Similarly, for any $i \in S_2$ it holds that

$C_i(\pi, \varphi) = |\{j \in N \backslash S : \varphi(j) < \varphi(i)\}| + \varphi^S(i)$ and
$C_i(\hat{\sigma}, \hat{\tau}) = |\{j \in N \backslash S : \varphi(j) < \varphi(i)\}| + \hat{\tau}^S(i)$.

Hence, by definition of the game v,

$$\begin{aligned} v(S) &= \sum_{i \in S} (C_i(\pi, \varphi) - C_i(\hat{\sigma}, \hat{\tau})) \\ &= \sum_{i \in S_1} (C_i(\pi, \varphi) - C_i(\hat{\sigma}, \hat{\tau})) + \sum_{i \in S_2} (C_i(\pi, \varphi) - C_i(\hat{\sigma}, \hat{\tau})) \\ &= \sum_{i \in S_1} (\pi^S(i) - \hat{\sigma}^S(i)) + \sum_{i \in S_2} (\varphi^S(i) - \hat{\tau}^S(i)) \\ &= \sum_{i \in S_1} \pi^S(i) - \sum_{i \in S_1} \hat{\sigma}^S(i) + \sum_{i \in S_2} \varphi^S(i) - \sum_{i \in S_2} \hat{\tau}^S(i). \end{aligned}$$

Clearly, for all $i \in S_1$ it holds that $C_i(\pi^S, \varphi^S) \geq \pi^S(i)$ and for all $i \in S_2$ it holds that $C_i(\pi^S, \varphi^S) \geq \varphi^S(i)$. Hence, by definition of the game w and (2),

$$\begin{aligned} w(S) &= \sum_{i\in S} C_i(\pi^S,\varphi^S) - (1+2+...+|S|) \\ &= \sum_{i\in S_1} C_i(\pi^S,\varphi^S) + \sum_{i\in S_2} C_i(\pi^S,\varphi^S) - (1+2+...+|S|) \\ &\geq \sum_{i\in S_1} \pi^S(i) + \sum_{i\in S_2} \varphi^S(i) - (1+2+...+|S|). \end{aligned}$$

It remains to prove that,

$$\begin{aligned} w(S)-v(S) &\geq \sum_{i\in S_1} \hat{\sigma}^S(i) + \sum_{i\in S_2} \hat{\tau}^S(i) - (1+2+...+|S|) \\ &\geq 0. \end{aligned}$$

In other words,

$$\sum_{i\in S_1} \hat{\sigma}^S(i) + \sum_{i\in S_2} \hat{\tau}^S(i) \geq 1+2+...+|S|.$$

We distinguish between two cases:

Case 1: There are no $i_1 \in S_1$ and $i_2 \in S_2$ with $\hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$.
Then we have $\sum_{i\in S_1} \hat{\sigma}^S(i) + \sum_{i\in S_2} \hat{\tau}^S(i) = 1+2+...+|S|$.

Case 2: There is a pair of players $i_1 \in S_1$ and $i_2 \in S_2$ with $\hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$.
We classify the positions $1,2,...,|S|$ of the scheme $(\hat{\sigma}^S,\hat{\tau}^S)$ in the following four classes.

$F := \{k \in \{1,...,|S|\} : k = \hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$ for some $i_1 \in S_1$ and $i_2 \in S_2\}$
$E := \{k \in \{1,...,|S|\} : k = \hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$ for some $i_1 \in S_2$ and $i_2 \in S_1\}$
$M_1 := \{k \in \{1,...,|S|\} : k = \hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$ for some $i_1 \in S_1$ and $i_2 \in S_1\}$
$M_2 := \{k \in \{1,...,|S|\} : k = \hat{\sigma}^S(i_1) = \hat{\tau}^S(i_2)$ for some $i_1 \in S_2$ and $i_2 \in S_2\}$.

First we observe that $|F| = |E|$. This holds since,
if $|F| > |E|$ then $\left|\{\hat{\sigma}^S(i_1) : i_1 \in S_1\}\right| + \left|\{\hat{\tau}^S(i_2) : i_2 \in S_2\}\right| > |S|$, and
if $|F| < |E|$ then $\left|\{\hat{\sigma}^S(i_1) : i_1 \in S_1\}\right| + \left|\{\hat{\tau}^S(i_2) : i_2 \in S_2\}\right| < |S|$.
Which both are contradictions, since S_1 and S_2 form a partition of S.

We define for a position $s \in \{1,...,|S|\}$ the following counters:
$F(s) := \{s,...,|S|\} \cap F$ and
$E(s) := \{s,...,|S|\} \cap E$.

To prove that $\sum_{i\in S_1} \hat{\sigma}^S(i) + \sum_{i\in S_2} \hat{\tau}^S(i) \geq 1+2+...+|S|$ it is enough to show that

$$\begin{aligned} \sum_{i\in S_1} \hat{\sigma}^S(i) + \sum_{i\in S_2} \hat{\tau}^S(i) &= \sum_{k\in M_1} k + \sum_{k\in M_2} \sum k + \sum_{k\in F} 2k \\ &\geq \sum_{k\in M_1} k + \sum_{k\in M_2} k + \sum_{k\in F} k + \sum_{k\in E} k \\ &= 1+2+...+|S| \end{aligned}$$

where the two equalities are trivial and the inequality is true if the following claim holds.

Claim: $F(s) \geq E(s)$ for all $s \in \{1,...,|S|\}$

Proof of the claim:

Suppose the contrary. Then there is a $t \in \{1,...,|S|\}$ with $F(t) < E(t)$. Let $s^* := \max\{t \in \{1,...,|S|\} : F(t) < E(t)\}$. It is easy to verify that $F(s^*+1) = E(s^*+1)$. Let $m = F(s^*+1)$. Let $T := \{s^*+1,...,|S|\}$ be the positions from s^*+1 to $|S|$ in the scheme $(\hat{\sigma}^S,\hat{\tau}^S)$. Let $p = |T \cap M_1|$ and $q = |T \cap M_2|$.

It is easy to verify that, $|T \cap (M_2 \cup E)| = m+q = |T \cap (M_2 \cup F)|$. So, since $s^* \in E$ it follows that,

$|(T \cup \{s^*\}) \cap (M_2 \cup E)| = m + q + 1$ and
$|(T \cup \{s^*\}) \cap (M_2 \cup F)| = m + q$.

Hence, there is a player $i \in S_2$ with,
$\hat{\sigma}^S(i) \in (T \cup \{s^*\}) \cap (M_2 \cup E)$ and
$\hat{\tau}^S(i) \notin (T \cup \{s^*\}) \cap (M_2 \cup F)$.

It is easy to check that

$$\hat{\tau}^S(i) < s^* \leq \hat{\sigma}^S(i). \tag{3}$$

Using the same argument, replacing M_2 by M_1,
$|(T \cup \{s^*\}) \cap (M_1 \cup E)| = m + p + 1$ and
$|(T \cup \{s^*\}) \cap (M_1 \cup F)| = m + p$.

Hence, there is a player $j \in S_1$ with,
$\hat{\sigma}^S(j) \notin (T \cup \{s^*\}) \cap (M_1 \cup F)$ and
$\hat{\tau}^S(j) \in (T \cup \{s^*\}) \cap (M_1 \cup E)$.

Obviously $i \neq j$. It is easy to check that player j holds

$$\hat{\sigma}^S(j) < s^* \leq \hat{\tau}^S(j). \tag{4}$$

Now, note that,

$$\hat{\tau}(i) \geq \hat{\sigma}(i) > \hat{\sigma}(j) \geq \hat{\tau}(j), \tag{5}$$

where the first inequality follows from $i \in S_2$. The second inequality holds since (3) and (4) imply $\hat{\sigma}^S(i) \geq s^* > \hat{\sigma}^S(j)$. The third inequality follows from $j \in S_1$.

Note that (3) and (4) imply $\hat{\tau}^S(j) \geq s^* > \hat{\tau}^S(i)$, which contradicts (5). This completes the proof of the claim.

Hence, $F(s) \geq E(s)$ for all $s \in \{1, ..., |S|\}$ and consequently $\sum_{k \in F} k \geq \sum_{k \in E} k$ which implies that $\sum_{i \in S_1} \hat{\sigma}^S(i) + \sum_{i \in S_2} \hat{\tau}^S(i) \geq 1 + 2 + ... + |S|$. This completes the proof of the theorem. □

Now we have shown that for simple 2−PS situations, $(M, N, (\pi, \varphi))$ it holds that $w \geq v$ and $w(N) = v(N)$. Consequently, $C(w) \subseteq C(v)$. We are going to prove the balancedness of the 2−PS game v arising from a simple 2−PS situation, by showing that a specific marginal vector of the optimistic 2−PS game w lies in the core of w and hence in the core of v. More precisely, the marginal vector of w with respect to the initial order on the first machine.

Let (N, v) be a TU game and $\sigma \in \prod(N)$ be an order for N, then the vector $m^\sigma(v) \in \mathbb{R}^N$ defined by

$$m_i^\sigma(v) := v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma)) \text{ for all } i \in N$$

is the marginal vector of v with respect to σ. The number $m_i^\sigma(v)$ is the marginal contribution of player $i \in N$ with respect to σ.

For any coalition $S \subseteq N$, the vector $m^{\sigma,S}(v) \in \mathbb{R}^S$ defined by

$$m_i^{\sigma,S}(v) := v((P_i(\sigma) \cap S) \cup \{i\}) - v(P_i(\sigma) \cap S) \text{ for all } i \in S$$

is the marginal vector of v with respect to σ and coalition S. The number $m_i^{\sigma,S}(v)$ is the marginal contribution of player $i \in S$ to coalition S with respect to σ. Notice that $m^{\sigma,N}(v) = m^\sigma(v)$.

Clearly $\sum\limits_{i \in S} m_i^{\sigma,S}(v) = v(S)$, which follows directly from the observation that $v(\emptyset) = 0$ and that for any two players $i, j \in S$, with j following i with respect to σ, i.e., j is the next player of S after i with respect to σ, it holds that $(P_i(\sigma) \cap S) \cup \{i\} = P_j(\sigma) \cap S$.

We introduce some useful sets that will be of great help in proving the balancedness of the optimistic game w arising from a simple 2−PS situation:

$PF_i^S(\pi, \varphi) := \{j \in S : \pi(j) < \pi(i) \text{ and } \varphi(j) > \varphi(i)\}$,

$FP_i^S(\pi, \varphi) := \{j \in S : \pi(j) > \pi(i) \text{ and } \varphi(j) < \varphi(i)\}$.

For any $i \in S \subseteq N$, the set $PF_i^S(\pi, \varphi)$ represents the players $j \in S$ that are predecessors of i with respect to the order π and followers of i with respect to the order φ.

Similarly, for any $i \in S \subseteq N$, the set $FP_i^S(\pi, \varphi)$ represents the players $j \in S$ that are followers of i with respect to the order π and predecessors of i with respect to the order φ.

It is easy to check that for any $i \in S \subseteq N$ it holds

$$\varphi^S(i) = \pi^S(i) + \left|FP_i^S(\pi, \varphi)\right| - \left|PF_i^S(\pi, \varphi)\right|. \tag{6}$$

The next example clarifies relation (6).

Example 4.1 Consider the simple 2−PS situation with:

$N = \{1, 2, 3, 4, 5\}$, and the initial scheme given by

π	1	2	3	4	5
φ	4	3	5	2	1

Take $S = \{1, 2, 3, 5\}$ and $i = 2$, then we have

π^S	1	2	3	5
φ^S	3	5	2	1

and $PF_i^S(\pi, \varphi) = 1$, $FP_i^S(\pi, \varphi) = \{3, 5\}$ which leads to
$\varphi^S(i) = 3 = 2 + 2 - 1 = \pi^S(i) + \left|FP_i^S(\pi, \varphi)\right| - \left|PF_i^S(\pi, \varphi)\right|$. $\triangle$

Finally, we introduce the set

$$M_i^S(\pi, \varphi) := \left\{j \in PF_i^S(\pi, \varphi) : \varphi^{S_i}(j) \geq \pi^{S_i}(j)\right\} \tag{7}$$

where $S_i = P_i(\pi) \cap S$.

Now, we can prove the balancedness of an optimistic 2-PS game.

Theorem 4.2 *Let (N, w) be the optimistic game arising from a simple 2−PS situation $(M, N, (\pi, \varphi))$. Then it holds that $m^{\pi,N}(w) \in C(w)$.*

Proof. Clearly $\sum\limits_{i \in N} m_i^{\pi,N}(w) = w(N)$. Let $S \subset N$. Our aim is prove that

$$\sum_{i \in S} m_i^{\pi,N}(w) \geq w(S) = \sum_{i \in S} m_i^{\pi,S}(w). \tag{8}$$

Let us rewrite the left hand side of the inequality (8).

$$\begin{aligned}\sum_{i\in S} m_i^{\pi,N}(w) &= \sum_{i\in S}\left|M_i^N(\pi,\varphi)\right| \\ &= \sum_{k\in N}\left|\left\{i\in S : k\in M_i^N(\pi,\varphi)\right\}\right| \\ &\geq \sum_{k\in N}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} \\ &= \sum_{k\in S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} \\ &+ \sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\}.\end{aligned}$$

Here the first equality follows from lemma 6.2. The second equality is trivial. The first inequality follows from lemma 6.4. The third equality is trivial. Hence,

$$\begin{aligned}\sum_{i\in S} m_i^{\pi,N}(w) &\geq \sum_{k\in S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} \\ &+ \sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\}.\end{aligned} \tag{9}$$

Let us now rewrite the right hand side of the inequality (8).

$$\begin{aligned}\sum_{i\in S} m_i^{\pi,S}(w) &= \sum_{i\in S}\left|M_i^S(\pi,\varphi)\right| \\ &= \sum_{k\in S}\left|\left\{i\in S : k\in M_i^S(\pi,\varphi)\right\}\right| \\ &= \sum_{k\in S}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\}.\end{aligned}$$

Here the first equality follows from lemma 6.2. The second equality is trivial. The third equality follows from lemma 6.3. Hence,

$$\sum_{i\in S} m_i^{\pi,S}(w) = \sum_{k\in S}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\}. \tag{10}$$

In order to show that (8) holds it is enough to show that $\sum_{i\in S} m_i^{\pi,N}(w) - \sum_{i\in S} m_i^{\pi,S}(w) \geq 0$, and from (9) and (10) it is enough to show that,

$$\begin{aligned}\sum_{k\in S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} &+ \sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} \\ -\sum_{k\in S}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\} &\geq 0.\end{aligned} \tag{11}$$

Clearly, if there is no $k\in S$ that holds (30) we are done with the proof. If this is not the case, let us define the following partition of S,

$S_1 := \{i\in S : i \text{ holds } (30)\}$,

$S_2 := \{i\in S : i \text{ does not hold } (30)\}$.

Since S_1 and S_2 form a partition of S we can write (11) as follows:

$$\begin{aligned}\sum_{k\in S_1}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} &- \sum_{k\in S_1}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\} \\ +\sum_{k\in S_2}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} &- \sum_{k\in S_2}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\} \\ +\sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} &\geq 0.\end{aligned}$$

By definition of S_2, for any $k\in S_2$ it holds that, $\sum_{k\in S_2}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\} \geq \sum_{k\in S_2}\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\}$.

Hence, to show that (11) holds, it is enough to check that

$$\begin{aligned}&\sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\}\geq\\&\geq\sum_{k\in S_1}\left[\max\left\{\varphi^{S}(k)-\pi^{S}(k),0\right\}-\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\}\right].\end{aligned}\tag{12}$$

Let us focus on the left hand side of the inequality (12):

$$\begin{aligned}&\sum_{k\in N\setminus S}\max\left\{\varphi^{P_k(\pi)\cup S}(k)-\pi^{P_k(\pi)\cup S}(k),0\right\}\\&\geq\sum_{k\in N\setminus S}|\{i\in S:k\in G_i\}|\\&=\sum_{i\in S_1}|G_i|\\&=\sum_{i\in S_1}\left[\max\left\{\varphi^{S}(i)-\pi^{S}(i),0\right\}-\max\left\{\varphi^{P_i(\pi)\cup S}(i)-\pi^{P_i(\pi)\cup S}(i),0\right\}\right]\end{aligned}$$

Here the first inequality follows from lemma 6.6. The first equality follows from the definition of G_i. The second equality follows from lemma 6.5. This completes the proof.

Hence, $\sum_{i\in S} m_i^{\pi,N}(w)-\sum_{i\in S} m_i^{\pi,S}(w)\geq 0$ and $m^{\pi,N}(w)\in C(w)$. □

Theorem 4.3 *The marginal vector $m^{\pi,N}(w)$ is in the core of the $2-PS$ game (N,v) arising from any simple $2-PS$ situation.*

Proof. From theorem 4.1 we have that $w(S)\geq v(S)$ for all $S\subset N$, and $w(N)=v(N)$. Then $C(w)\subseteq C(v)$. From theorem 4.2 we have that $m^{\pi,N}(w)\in C(w)$. Consequently, $m^{\pi,N}(w)\in C(v)$. □

Corollary 4.1 *The marginal vector $m^{\varphi,N}(w)$, associated to the initial order on the second machine φ, lies in the core of the $2-PS$ game (N,v) arising from any simple $2-PS$ situation.*

Proof. It is straightforward by switching the names of the machines. □

5 Final remarks

Some of the results presented in Section 2 and 3 for 2−PS situations can easily be generalized for $m-$PS situations, i.e., 5−tuples $(M,N,(\pi^k)_{k\in M},(\alpha_i)_{i\in N},(p_i^k)_{\substack{i\in N\\k\in M}})$, where $M=\{1,...,m\}$ is a finite set of m machines. Now, each agent has m jobs to be processed, one on each machine. It is readily seen that both Proposition 2.1 and Proposition 2.2 can be generalized to $m-$PS situations.

In section 3 we have introduced the 2−PS game v arising from a 2−PS situation. The $m-$PS game v is also monotonic and superadditive. Moreover, proposition 3.1 can be generalized as well. The optimistic $m-$PS game w is easily derived for $m-$PS situations, and by the new proposition 2.2, the worth of a coalition $S\subset N$ can still easily be described.

We conclude with an open question: is $w(S)\geq v(S)$ for any $S\subset N$?. It may be of help in proving the balancedness of (N,v), which is also an open problem.

6 Appendix

This Appendix contains some technical Lemmata and some results that follow from these Lemmata that are needed to prove Theorem 4.1 and 4.2.

Lemma 6.1 describes the existence of an optimal scheme of a coalition that satisfies some monotony condition on the completion times.

Lemma 6.1 *For a simple* $2-PS$ *situation* $(M, N, (\pi, \varphi))$ *it holds that for all* $S \subseteq N$ *there is an optimal scheme* $(\hat{\sigma}, \hat{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ *such that for all* $i \in S$,
if $\pi(i) \geq \varphi(i)$ *then* $\hat{\sigma}(i) \geq \hat{\tau}(i)$ *and*
if $\pi(i) \leq \varphi(i)$ *then* $\hat{\sigma}(i) \leq \hat{\tau}(i)$.

Proof. Take $S \subseteq N$.

On the contrary, suppose that for all optimal schemes $(\hat{\sigma}, \hat{\tau})$ for S there exists a player $i \in S$ such that either:

$$\pi(i) > \varphi(i) \text{ and } \hat{\sigma}(i) < \hat{\tau}(i) \text{ or} \tag{13}$$

$$\pi(i) < \varphi(i) \text{ and } \hat{\sigma}(i) > \hat{\tau}(i) \ . \tag{14}$$

Given an optimal scheme for S, let us call players $i \in S$ for which (13) or (14) is satisfied *strange*. Choose an optimal scheme $(\hat{\sigma}, \hat{\tau})$ for S such that the number of strange players is minimal. We construct a new optimal scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ for S with one strange player less, which thus establishes a contradiction.

Let i be a strange player in S, without loss of generality, we may assume that i satisfies (13).

Let $T_1 \in S/\pi$ be such that $i \in T_1$ and $T_2 \in S/\varphi$ be such that $i \in T_2$. Clearly T_1 and T_2 have to *overlap* with respect to π and φ, i.e., $\pi(T_1) \cap \varphi(T_2) \neq \emptyset$, where $\pi(T_1) = \{\pi(j) : j \in T_1\}$ and $\varphi(T_2) = \{\varphi(j) : j \in T_2\}$. Otherwise i could not be strange. Note that $S/\pi = S/\hat{\sigma}$, $S/\varphi = S/\hat{\tau}$ and $\pi(T_1) = \hat{\sigma}(T_1)$, $\varphi(T_2) = \hat{\tau}(T_2)$.

We distinguish between two cases:

I) $\max\limits_{j \in T_1} \pi(j) \geq \max\limits_{j \in T_2} \varphi(j)$.

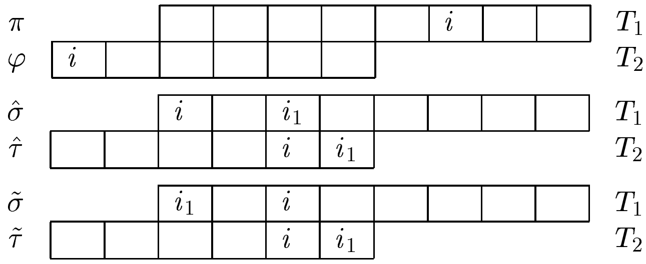
Since i satisfies (13) it holds that $\hat{\tau}(i) \in \pi(T_1) \cap \varphi(T_2)$. Let $i_1 \in T_1$ be such that $\hat{\sigma}(i_1) = \hat{\tau}(i)$. Note that $i_1 \neq i$.

We construct the new scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ defined by, $\tilde{\tau} := \hat{\tau}$ and

$$\tilde{\sigma}(j) := \begin{cases} \hat{\sigma}(j) & \text{if } \ j \in S \backslash \{i, i_1\} \\ \hat{\sigma}(i_1) & \text{if } \ j = i \\ \hat{\sigma}(i) & \text{if } \ j = i_1 \end{cases} .$$

It is easy to check that $C_j(\tilde{\sigma}, \tilde{\tau}) \leq C_j(\hat{\sigma}, \hat{\tau})$ for all $j \in S$. Hence, $C_j(\tilde{\sigma}, \tilde{\tau}) = C_j(\hat{\sigma}, \hat{\tau})$ for all $j \in S$ since $(\hat{\sigma}, \hat{\tau})$ is an optimal scheme for S. Note that i is not strange anymore. Moreover, if i_1 was not strange with respect to $(\hat{\sigma}, \hat{\tau})$, it still is not, if it was strange, it is still with respect to $(\tilde{\sigma}, \tilde{\tau})$. This follows from
$\tilde{\tau}(i_1) = \hat{\tau}(i_1) > \hat{\sigma}(i_1) = \hat{\tau}(i) > \hat{\sigma}(i) = \tilde{\sigma}(i_1)$ (the first inequality follows from $C_{i_1}(\tilde{\sigma}, \tilde{\tau}) = C_{i_1}(\hat{\sigma}, \hat{\tau})$ and $\tilde{\tau} = \hat{\tau}$). Furthermore, any other player in S remains unchanged.

So, $(\tilde{\sigma}, \tilde{\tau})$ is an optimal scheme for S with one strange player less. We arrive at a contradiction. Graphically:



II) $\max\limits_{j\in T_1} \pi(j) < \max\limits_{j\in T_2} \varphi(j)$.

We distinguish among three subcases:

A) If $\hat{\tau}(i) \in \pi(T_1) \cap \varphi(T_2)$.

We construct the new scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ as in case I) and $(\tilde{\sigma}, \tilde{\tau})$ is an optimal scheme for S with one strange player less. We arrive at a contradiction.

B) If $\hat{\tau}(i) \notin \pi(T_1) \cap \varphi(T_2)$, but $\hat{\sigma}(i) \in \pi(T_1) \cap \varphi(T_2)$.

Let $i_2 \in T_2$ be such that $\hat{\sigma}(i) = \hat{\tau}(i_2)$. Note that $i_2 \neq i$.

We construct the new scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ defined by, $\tilde{\sigma} := \hat{\sigma}$ and

$$\tilde{\tau}(j) := \begin{cases} \hat{\tau}(j) & \text{if} \quad j \in S\backslash \{i, i_2\} \\ \hat{\tau}(i_2) & \text{if} \quad j = i \\ \hat{\tau}(i) & \text{if} \quad j = i_2 \end{cases}.$$

Then clearly:

$C_i(\tilde{\sigma}, \tilde{\tau}) = C_i(\hat{\sigma}, \hat{\tau}) - [\hat{\tau}(i) - \hat{\tau}(i_2)]$,

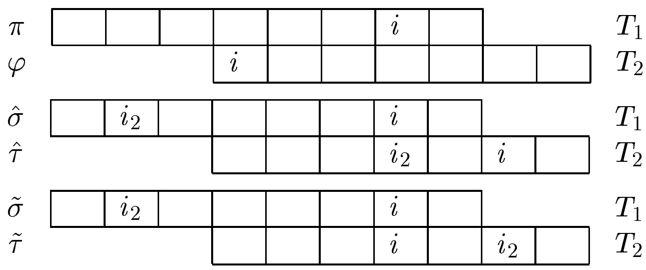
$C_{i_2}(\tilde{\sigma}, \tilde{\tau}) \leq C_{i_2}(\hat{\sigma}, \hat{\tau}) + [\hat{\tau}(i) - \hat{\tau}(i_2)]$ and,

$C_j(\tilde{\sigma}, \tilde{\tau}) = C_j(\hat{\sigma}, \hat{\tau})$ for all $j \in S\backslash \{i, i_2\}$.

Note that $\hat{\tau}(i) > \hat{\sigma}(i) = \hat{\tau}(i_2)$. So, $\hat{\tau}(i) - \hat{\tau}(i_2) > 0$. In fact, $C_{i_2}(\tilde{\sigma}, \tilde{\tau}) = C_{i_2}(\hat{\sigma}, \hat{\tau}) + [\hat{\tau}(i) - \hat{\tau}(i_2)]$ since $(\hat{\sigma}, \hat{\tau})$ is an optimal scheme for S.

Note that i is not strange anymore. Moreover, if i_2 was not strange with respect to $(\hat{\sigma}, \hat{\tau})$, it still is not, if i_2 was strange, it is still with respect to $(\tilde{\sigma}, \tilde{\tau})$. This follows from $\tilde{\tau}(i_2) = \hat{\tau}(i) > \hat{\sigma}(i) = \hat{\tau}(i_2) > \hat{\sigma}(i_2) = \tilde{\sigma}(i_2)$ (the second inequality follows from $C_{i_2}(\tilde{\sigma}, \tilde{\tau}) = C_{i_2}(\hat{\sigma}, \hat{\tau}) + [\hat{\tau}(i) - \hat{\tau}(i_2)]$ and $\tilde{\sigma} = \hat{\sigma}$). Furthermore, any other player in S remains unchanged.

So, $(\tilde{\sigma}, \tilde{\tau})$ is an optimal scheme for S with one strange player less. We arrive at a contradiction. Graphically:



C) If $\hat{\tau}(i) \notin \pi(T_1) \cap \varphi(T_2)$ and $\hat{\sigma}(i) \notin \pi(T_1) \cap \varphi(T_2)$.

Since i satisfies (13) we have that,

$\min\limits_{j\in T_1} \pi(j) = \min\limits_{j\in T_1} \hat{\sigma}(j) \leq \hat{\sigma}(i) < \min\limits_{j\in T_2} \hat{\tau}(j) = \min\limits_{j\in T_2} \varphi(j)$.

All inequalities are trivial except for $\hat{\sigma}(i) < \min\limits_{j\in T_2} \hat{\tau}(j)$. Let us prove this inequality.

Suppose the contrary, $\hat{\sigma}(i) \geq \min\limits_{j\in T_2} \hat{\tau}(j)$. By assumption of case II), i.e.,

$\max\limits_{j\in T_1} \pi(j) < \max\limits_{j\in T_2} \varphi(j)$, it follows that $\hat{\sigma}(i) \in \hat{\sigma}(T_1) \cap \hat{\tau}(T_2) = \pi(T_1) \cap \varphi(T_2)$, which contradicts assumption C).

Observe that from $\min\limits_{j\in T_1} \pi(j) < \min\limits_{j\in T_2} \varphi(j)$ and $\max\limits_{j\in T_1} \pi(j) < \max\limits_{j\in T_2} \varphi(j)$ it follows that for $j \in T_1 \cap T_2$ with $\pi(j) > \varphi(j)$ it holds that $\pi(j), \varphi(j) \in \pi(T_1) \cap \varphi(T_2)$. From this observation it follows that,

$$|\{j \in T_2 : j \in T_1 \text{ and } \pi(j) > \varphi(j)\}| \leq |\pi(T_1) \cap \varphi(T_2)| - 1. \tag{15}$$

In view of (15), it follows from $|\{j \in T_2 : \hat{\tau}(j) \in \pi(T_1) \cap \varphi(T_2)\}| = |\pi(T_1) \cap \varphi(T_2)|$ that there is a player $i_2 \in T_2$ with $\hat{\tau}(i_2) \in \pi(T_1) \cap \varphi(T_2)$ such that either $i_2 \notin T_1$ or $i_2 \in T_1$ and $\pi(i_2) \leq \varphi(i_2)$. Let $i_1 \in T_1$ be such that $\hat{\sigma}(i_1) = \hat{\tau}(i_2)$, with possibly $i_1 = i_2$. Note that $i \neq i_1$ and $i \neq i_2$.

We construct the new scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ from $(\hat{\sigma}, \hat{\tau})$, by switching i and i_1 on machine 1 and i and i_2 on machine 2:

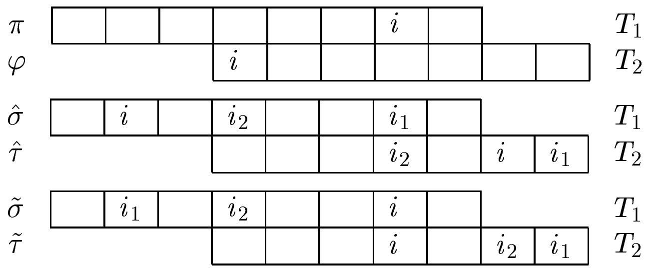
$$\tilde{\sigma}(j) := \begin{cases} \hat{\sigma}(j) & \text{if } \; j \in S \backslash \{i, i_1\} \\ \hat{\sigma}(i_1) & \text{if } \quad j = i \\ \hat{\sigma}(i) & \text{if } \quad j = i_1 \end{cases}, \text{ and } \tilde{\tau}(j) := \begin{cases} \hat{\tau}(j) & \text{if } \; j \in S \backslash \{i, i_2\} \\ \hat{\tau}(i_2) & \text{if } \quad j = i \\ \hat{\tau}(i) & \text{if } \quad j = i_2 \end{cases}.$$

Note that i is not strange anymore.

We distinguish two subsubcases:

a) $i_1 \neq i_2$.

From the arguments in I) and II)B), the new scheme $(\tilde{\sigma}, \tilde{\tau})$ is optimal and there are no new strange players. So, $(\tilde{\sigma}, \tilde{\tau})$ is an optimal scheme for S with one strange player less. We arrive at a contradiction. Graphically:

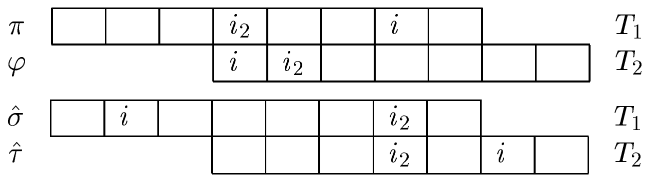


b) $i_1 = i_2$. So, $i_2 \in T_1$.

We construct a new scheme $(\tilde{\sigma}, \tilde{\tau}) \in \mathcal{A}^1(S) \times \mathcal{A}^2(S)$ from $(\hat{\sigma}, \hat{\tau})$ by switching i_2 and i on both machines. Then $\tilde{\tau}(i_2) = \hat{\tau}(i) > \hat{\sigma}(i) = \tilde{\sigma}(i_2)$.

We have chosen $i_2 \in T_2$ in such a way that either $i_2 \notin T_1$ or $i_2 \in T_1$ and $\pi(i_2) \leq \varphi(i_2)$. Since $i_2 \in T_1$, it holds that $\pi(i_2) \leq \varphi(i_2)$. Hence, i_2 is not a strange player with respect to $(\tilde{\sigma}, \tilde{\tau})$. Furthermore, any other player in S remains unchanged. Hence, there are no new strange players. Moreover, the final completion time of i_2 increases with the same amount as the final completion time of i decreases.

So, $(\tilde{\sigma}, \tilde{\tau})$ is an optimal scheme for S with one strange player less. We arrive at a contradiction. Graphically:



$\tilde{\sigma}$		i_2					i				T_1
$\tilde{\tau}$							i		i_2		T_2

$\square$ The following Lemma provides a relation between $m_i^{\pi,S}(w)$ and $M_i^S(\pi,\varphi)$.

Lemma 6.2 *For all $i \in S \subseteq N$ it holds that $m_i^{\pi,S}(w) = \left|M_i^S(\pi,\varphi)\right|$.*

Proof. Let i be a player in S and π the initial order on the first machine, then

$$\begin{aligned}
m_i^{\pi,S}(w) &= w((P_i(\pi)\cap S)\cup\{i\}) - w(P_i(\pi)\cap S) \\
&= \sum_{j\in S_i\cup\{i\}} C_j(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) - (1+...+|S_i\cup\{i\}|) \\
&\quad - \left(\sum_{j\in S_i} C_j(\pi^{S_i},\varphi^{S_i}) - (1+...+|S_i|)\right) \\
&= C_i(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) - |S_i\cup\{i\}| \\
&\quad + \sum_{j\in S_i} C_j(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) - \sum_{j\in S_i} C_j(\pi^{S_i},\varphi^{S_i}) \\
&= \sum_{j\in S_i}\left[C_j(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) - C_j(\pi^{S_i},\varphi^{S_i})\right] \\
&= \left|\left\{\begin{array}{r} j\in S : \pi(j)<\pi(i) \\ \varphi(j)>\varphi(i) \\ \varphi^{S_i}(j)\geq\pi^{S_i}(j) \end{array}\right\}\right| \\
&= \left|M_i^S(\pi,\varphi)\right|.
\end{aligned}$$

The second equality follows from (2). The fourth equality follows since i is the last player of $S_i\cup\{i\}$ with respect to π and hence $C_i(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) = |S_i\cup\{i\}|$.

The fifth equality follows from, $C_j(\varphi^{S_i\cup\{i\}}) = C_j(\varphi^{S_i})$ for all $j\in S_i$ such that $\varphi(j)<\varphi(i)$ and $C_j(\varphi^{S_i\cup\{i\}}) = C_j(\varphi^{S_i})+1$ for all $j\in S_i$ such that $\varphi(j)>\varphi(i)$. Consequently, $C_j(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) = C_j(\pi^{S_i},\varphi^{S_i})+1$ for all $j\in S_i$ such that $\varphi(j)>\varphi(i)$ and $\varphi^{S_i}(j)\geq\pi^{S_i}(j)$. And $C_j(\pi^{S_i\cup\{i\}},\varphi^{S_i\cup\{i\}}) = C_j(\pi^{S_i},\varphi^{S_i})$ for any other player $j\in S_i$. $\square$

In the following lemma we will calculate how many times a player $k\in S$ is in the sets $\left\{M_i^S(\pi,\varphi) : i\in S\right\}$.

Lemma 6.3 *For all $k\in S\subseteq N$ it holds that*
$\left|\left\{i\in S : k\in M_i^S(\pi,\varphi)\right\}\right| = \max\left\{\varphi^S(k)-\pi^S(k),0\right\}$.

Proof. First, note that for a player $i\in S$, $k\in PF_i^S(\pi,\varphi)$ if and only if $\pi(k)<\pi(i)$ and $\varphi(k)>\varphi(i)$ which is true if and only if $i\in FP_k^S(\pi,\varphi)$.

From this and (7) we can write for the left hand side of the equality.

$$\begin{aligned}
\left|\left\{i\in S : k\in M_i^S(\pi,\varphi)\right\}\right| &= \left|\left\{i\in S : k\in PF_i^S(\pi,\varphi) \text{ and } \varphi^{S_i}(k)\geq\pi^{S_i}(k)\right\}\right| \\
&= \left|\left\{i\in FP_k^S(\pi,\varphi) : \varphi^{S_i}(k)\geq\pi^{S_i}(k)\right\}\right|.
\end{aligned}$$

Hence,

$$\left|\left\{i\in S : k\in M_i^S(\pi,\varphi)\right\}\right| = \left|\left\{i\in FP_k^S(\pi,\varphi) : \varphi^{S_i}(k)\geq\pi^{S_i}(k)\right\}\right|. \tag{16}$$

Let us now calculate $\left|FP_k^S(\pi,\varphi)\right|$. Applying relation (6) to player k in S and in $S_k\cup\{k\}$ we have that

$$\varphi^S(k) = \pi^S(k) + \left|FP_k^S(\pi,\varphi)\right| - \left|PF_k^S(\pi,\varphi)\right| \text{ and} \tag{17}$$

$$\varphi^{S_k\cup\{k\}}(k) = \pi^{S_k\cup\{k\}}(k) + \left|FP_k^{S_k\cup\{k\}}(\pi,\varphi)\right| - \left|PF_k^{S_k\cup\{k\}}(\pi,\varphi)\right|. \tag{18}$$

Subtracting (17) − (18) gives

$$\begin{aligned}\varphi^S(k)-\pi^S(k)+\pi^{S_k\cup\{k\}}(k)-\varphi^{S_k\cup\{k\}}(k) &= \left|FP_k^S(\pi,\varphi)\right|-\left|FP_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &\quad -\left|PF_k^S(\pi,\varphi)\right|+\left|PF_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &= \left|FP_k^S(\pi,\varphi)\right|\\ &\quad -\left|PF_k^S(\pi,\varphi)\right|+\left|PF_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &= \left|FP_k^S(\pi,\varphi)\right|\end{aligned}$$

where the second equality follows from $FP_k^{S_k\cup\{k\}}(\pi,\varphi)=\emptyset$ since k is the last player in $S_k\cup\{k\}$ with respect to π. The third equality follows from the observation that $PF_k^{S_k\cup\{k\}}(\pi,\varphi)=PF_k^S(\pi,\varphi)$, since the set of predecessors of k with respect to π in $S_k\cup\{k\}$ is the same as in S. Hence,

$$\left|FP_k^S(\pi,\varphi)\right|=\varphi^S(k)-\pi^S(k)+\pi^{S_k\cup\{k\}}(k)-\varphi^{S_k\cup\{k\}}(k)\geq 0. \tag{19}$$

Now, we focus on the difference $\varphi^{S_i}(k)-\pi^{S_i}(k)$. Our aim is to show for which players i in $FP_k^S(\pi,\varphi)$ it is non-negative. Let $i\in FP_k^S(\pi,\varphi)$. Applying (6) to $k\in S_i$ yields

$$\varphi^{S_i}(k)=\pi^{S_i}(k)+\left|FP_k^{S_i}(\pi,\varphi)\right|-\left|PF_k^{S_i}(\pi,\varphi)\right|. \tag{20}$$

Subtracting (20) − (18) we get

$$\begin{aligned}\varphi^{S_i}(k)-\pi^{S_i}(k) &= \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)\\ &\quad+\left|FP_k^{S_i}(\pi,\varphi)\right|-\left|FP_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &\quad-\left|PF_k^{S_i}(\pi,\varphi)\right|+\left|PF_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &= \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)\\ &\quad+\left|FP_k^{S_i}(\pi,\varphi)\right|\\ &\quad-\left|PF_k^{S_i}(\pi,\varphi)\right|+\left|PF_k^{S_k\cup\{k\}}(\pi,\varphi)\right|\\ &= \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)\\ &\quad+\left|FP_k^{S_i}(\pi,\varphi)\right|\end{aligned}$$

where, using the same argument as before, the second equality follows from $FP_k^{S_k\cup\{k\}}(\pi,\varphi)=\emptyset$. The third equality follows from $PF_k^{S_k\cup\{k\}}(\pi,\varphi)=PF_k^{S_i}(\pi,\varphi)$. Note that $\pi(i)>\pi(k)$ and the set of predecessors of k with respect to π in $S_k\cup\{k\}$ is the same as in S_i. Hence,

$$\varphi^{S_i}(k)-\pi^{S_i}(k)=\varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)+\left|FP_k^{S_i}(\pi,\varphi)\right|. \tag{21}$$

We distinguish between two cases:

Case I: $\max\left\{\varphi^S(k)-\pi^S(k),0\right\}=0$.
From (19) and $\varphi^S(k)-\pi^S(k)\leq 0$ it follows that

$$\left|FP_k^S(\pi,\varphi)\right|\leq\pi^{S_k\cup\{k\}}(k)-\varphi^{S_k\cup\{k\}}(k). \tag{22}$$

Let $i\in FP_k^S(\pi,\varphi)$ then by (21) we have that

$$\begin{aligned}\varphi^{S_i}(k)-\pi^{S_i}(k) &= \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)+\left|FP_k^{S_i}(\pi,\varphi)\right|\\ &< \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)+\left|FP_k^{S}(\pi,\varphi)\right|\\ &\leq \varphi^{S_k\cup\{k\}}(k)-\pi^{S_k\cup\{k\}}(k)+\pi^{S_k\cup\{k\}}(k)-\varphi^{S_k\cup\{k\}}(k)\\ &= 0.\end{aligned}$$

Here the first inequality follows from $FP_k^{S_i}(\pi,\varphi) \subsetneqq FP_k^S(\pi,\varphi)$. Since $S_i \subseteq S$, and any follower of k with respect to π in S_i is also a follower of k with respect to π in S. Moreover, $i \in FP_k^S(\pi,\varphi)$, but $i \notin FP_k^{S_i}(\pi,\varphi)$. The second inequality follows from (22). Hence, $\{i \in FP_k^S(\pi,\varphi) : \varphi^{S_i}(k) \geq \pi^{S_i}(k)\} = \emptyset$.

So, from (16) it follows that
$\left|\{i \in S : k \in M_i^S(\pi,\varphi)\}\right| = \left|\{i \in FP_k^S(\pi,\varphi) : \varphi^{S_i}(k) \geq \pi^{S_i}(k)\}\right| = 0,$
which completes the proof of case I.

Case II: $\max\{\varphi^S(k) - \pi^S(k), 0\} > 0$.

Hence, $\varphi^S(k) - \pi^S(k) > 0$. Then from (19) and $\pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) \geq 0$,

$$\left|FP_k^S(\pi,\varphi)\right| = \varphi^S(k) - \pi^S(k) + \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) > 0 \tag{23}$$

Now, let us define a special player $i^* \in FP_k^S(\pi,\varphi)$. Let i^* be such that
$\left|\{i \in FP_k^S(\pi,\varphi) : \pi(i) < \pi(i^*)\}\right| = \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k)$.

Player i^* exists because $\pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) \geq 0$ since k is the last player in $S_k \cup \{k\}$ with respect to π. And in case II it holds that
$\left|FP_k^S(\pi,\varphi)\right| \geq \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k)$.

Let $i \in FP_k^S(\pi,\varphi)$ be such that $\pi(i^*) \leq \pi(i)$. By (21) it holds that

$$\begin{aligned} \varphi^{S_i}(k) - \pi^{S_i}(k) \quad &= \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \left|FP_k^{S_i}(\pi,\varphi)\right| \\ &\geq \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \left|FP_k^{S_{i^*}}(\pi,\varphi)\right| \\ &= \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) \\ &= 0 \end{aligned} \tag{24}$$

where the first inequality follows from the observation that $\pi(i^*) \leq \pi(i)$ which implies $FP_k^{S_{i^*}}(\pi,\varphi) \subseteq FP_k^{S_i}(\pi,\varphi)$. Since $S_{i^*} \subseteq S_i$, and any follower of k with respect to π in S_{i^*} is also a follower of k with respect to π in S_i. The second equality follows from the definition of i^* that clearly implies that

$$\left|FP_k^{S_{i^*}}(\pi,\varphi)\right| = \left|\{i \in FP_k^S(\pi,\varphi) : \pi(i) < \pi(i^*)\}\right| = \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k). \tag{25}$$

Now, let us consider $i \in FP_k^S(\pi,\varphi)$ be such that $\pi(i^*) > \pi(i)$. By (21) it holds that

$$\begin{aligned} \varphi^{S_i}(k) - \pi^{S_i}(k) \quad &= \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \left|FP_k^{S_i}(\pi,\varphi)\right| \\ &< \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \left|FP_k^{S_{i^*}}(\pi,\varphi)\right| \\ &= \varphi^{S_k\cup\{k\}}(k) - \pi^{S_k\cup\{k\}}(k) + \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) \\ &= 0 \end{aligned} \tag{26}$$

where the inequality follows from the observation that $\pi(i^*) > \pi(i)$ which implies $FP_k^{S_{i^*}}(\pi,\varphi) \supsetneqq FP_k^{S_i}(\pi,\varphi)$. Since $S_{i^*} \supsetneqq S_i$, and any follower of k with respect to π in S_i is also a follower of k with respect to π in S_{i^*}. Note that $i \in FP_k^{S_{i^*}}(\pi,\varphi)$, but $i \notin FP_k^{S_i}(\pi,\varphi)$. The second equality follows from (25).

Hence, for any $i \in FP_k^S(\pi,\varphi)$ with $\pi(i^*) \leq \pi(i)$ it holds that $\varphi^{S_i}(k) - \pi^{S_i}(k) \geq 0$. And for any $i \in FP_k^S(\pi,\varphi)$ with $\pi(i^*) > \pi(i)$ it holds that $\varphi^{S_i}(k) - \pi^{S_i}(k) < 0$. Which follows from (24) and (26), respectively. Consequently, by (16) we have

$$\begin{aligned}
\left|\left\{i \in S : k \in M_i^S(\pi,\varphi)\right\}\right| &= \left|\left\{i \in FP_k^S(\pi,\varphi) : \varphi^{S_i}(k) \geq \pi^{S_i}(k)\right\}\right| \\
&= \left|\left\{i \in FP_k^S(\pi,\varphi) : \pi(i) \geq \pi(i^*)\right\}\right| \\
&= \left|FP_k^S(\pi,\varphi)\right| - \left|\left\{i \in FP_k^S(\pi,\varphi) : \pi(i) < \pi(i^*)\right\}\right| \\
&= \varphi^S(k) - \pi^S(k) + \pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k) \\
&\quad - \left(\pi^{S_k\cup\{k\}}(k) - \varphi^{S_k\cup\{k\}}(k)\right) \\
&= \varphi^S(k) - \pi^S(k)
\end{aligned}$$

where the fourth equality follows from (23) and (25). This completes the proof of case II. □

As a consequence of Lemma 6.3 we can rewrite $w(S)$ for a coalition $S \subseteq N$:

$$\begin{aligned}
w(S) &= \sum_{i\in S} m_i^{\pi,S}(w) \\
&= \sum_{i\in S} \left|M_i^S(\pi,\varphi)\right| \\
&= \left|\left\{i \in S : k \in M_i^S(\pi,\varphi)\right\}\right| \\
&= \sum_{i\in S} \max\left\{\varphi^S(i) - \pi^S(i), 0\right\}.
\end{aligned}$$

The next lemma is a technical one that is needed to give an lower bound on some the sum of payoffs of the marginal vector in consideration.

Lemma 6.4 *For all $k \in N$ and $S \subseteq N$ it holds that*

$$\left|\left\{i \in S : k \in M_i^N(\pi,\varphi)\right\}\right| \geq \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}.$$

Proof. Let $k \in N$ and $S \subseteq N$.

Note that the left hand side of the inequality can be written as

$$\begin{aligned}
\left|\left\{i \in S : k \in M_i^N(\pi,\varphi)\right\}\right| &= \left|\left\{i \in S : k \in PF_i^N(\pi,\varphi) \text{ and } \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\right\}\right| \\
&= \left|\left\{i \in S : i \in FP_k^N(\pi,\varphi) \text{ and } \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\right\}\right| \\
&= \left|\left\{i \in S : i \in FP_k^S(\pi,\varphi) \text{ and } \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\right\}\right| \\
&= \left|\left\{i \in FP_k^S(\pi,\varphi) : \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\right\}\right|,
\end{aligned}$$

where the first equality follows from (7). The second equality follows from the fact that $k \in PF_i^N(\pi,\varphi)$ if and only if $i \in FP_k^N(\pi,\varphi)$. The third and fourth equalities are trivial. Hence,

$$\left|\left\{i \in S : k \in M_i^N(\pi,\varphi)\right\}\right| = \left|\left\{i \in FP_k^S(\pi,\varphi) : \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\right\}\right|. \tag{27}$$

By lemma 6.3, the right hand side of the inequality can be written as

$$\begin{aligned}
&\max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\} = \left|\left\{i \in P_k(\pi)\cup S : k \in M_i^{P_k(\pi)\cup S}(\pi,\varphi)\right\}\right| \\
&= \left|\left\{i \in P_k(\pi)\cup S : k \in PF_i^{P_k(\pi)\cup S}(\pi,\varphi) \text{ and } \varphi^{(P_k(\pi)\cup S)_i}(k) \geq \pi^{(P_k(\pi)\cup S)_i}(k)\right\}\right| \\
&= \left|\left\{i \in FP_k^{P_k(\pi)\cup S}(\pi,\varphi) : \varphi^{(P_k(\pi)\cup S)_i}(k) \geq \pi^{(P_k(\pi)\cup S)_i}(k)\right\}\right| \\
&= \left|\left\{i \in FP_k^S(\pi,\varphi) : \varphi^{(P_k(\pi)\cup S)_i}(k) \geq \pi^{(P_k(\pi)\cup S)_i}(k)\right\}\right|.
\end{aligned}$$

Here the first equality follows from lemma 6.3. The second equality from (7). The third equality from $k \in PF_i^{P_k(\pi)\cup S}(\pi,\varphi)$ if and only if $i \in FP_k^{P_k(\pi)\cup S}(\pi,\varphi)$. The last equality from the observation that $FP_k^{P_k(\pi)\cup S}(\pi,\varphi) = FP_k^S(\pi,\varphi)$ since the followers of k with respect to π in $P_k(\pi)\cup S$ are precisely the followers of k with respect to π in S. Hence,

$$\begin{aligned}
&\max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\} = \\
&= \left|\left\{i \in FP_k^S(\pi,\varphi) : \varphi^{(P_k(\pi)\cup S)_i}(k) \geq \pi^{(P_k(\pi)\cup S)_i}(k)\right\}\right|
\end{aligned} \tag{28}$$

We will now compare (27) and (28). The next expression will be useful. Let i be a player in $FP_k^S(\pi,\varphi)$. Applying relation (6) for player k in $P_i(\pi)$ yields

$$\begin{aligned} \varphi^{P_i(\pi)}(k) - \pi^{P_i(\pi)}(k) &= \left|FP_k^{P_i(\pi)}(\pi,\varphi)\right| - \left|PF_k^{P_i(\pi)}(\pi,\varphi)\right| \\ &\geq \left|FP_k^{(P_k(\pi)\cup S)_i}(\pi,\varphi)\right| - \left|PF_k^{(P_k(\pi)\cup S)_i}(\pi,\varphi)\right| \\ &= \varphi^{(P_k(\pi)\cup S)_i}(k) - \pi^{(P_k(\pi)\cup S)_i}(k) \end{aligned}$$

where the last equality also follows from (6) applied to player k in $(P_k(\pi)\cup S)_i$. The inequality follows from the observation that $PF_k^{P_i(\pi)}(\pi,\varphi) = PF_k^{(P_k(\pi)\cup S)_i}(\pi,\varphi)$, since the predecessors of k with respect to π in $(P_k(\pi)\cup S)_i$ are precisely the predecessors of k with respect to π in $P_i(\pi)$. And also from the observation that $FP_k^{P_i(\pi)}(\pi,\varphi) \supseteq FP_k^{(P_k(\pi)\cup S)_i}(\pi,\varphi)$ since $P_i(\pi) \supseteq P_i(\pi)\cap(P_k(\pi)\cup S) = (P_k(\pi)\cup S)_i$, and hence, any follower of k with respect to π in $(P_k(\pi)\cup S)_i$ is also a follower of k with respect to π in $P_i(\pi)$. Hence,

$$\varphi^{P_i(\pi)}(k) - \pi^{P_i(\pi)}(k) \geq \varphi^{(P_k(\pi)\cup S)_i}(k) - \pi^{(P_k(\pi)\cup S)_i}(k) \tag{29}$$

So, we can write

$$\begin{aligned} \left|\{i\in S : k\in M_i^N(\pi,\varphi\}\right| &= \left|\{i\in FP_k^S(\pi,\varphi) : \varphi^{P_i(\pi)}(k) \geq \pi^{P_i(\pi)}(k)\}\right| \\ &\geq \left|\{i\in FP_k^S(\pi,\varphi) : \varphi^{(P_k(\pi)\cup S)_i}(k) \geq \pi^{(P_k(\pi)\cup S)_i}(k)\}\right| \\ &= \max\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\} \end{aligned}$$

where the first equality follows from (27), the inequality follows from (29), and the final equality from (28) . □

Now, by lemma 6.2 and 6.4 we can write

$$\begin{aligned} \sum_{i\in S} m_i^{\pi,N}(w) &= \sum_{i\in S} \left|M_i^N(\pi,\varphi)\right| \\ &= \sum_{k\in N} \left|\{i\in S : k\in M_i^N(\pi,\varphi)\}\right| \\ &\geq \sum_{k\in N} \max\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\} \\ &= \sum_{k\in S} \max\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\} \\ &+ \sum_{k\in N\setminus S} \max\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\} \end{aligned}$$

Let us remember that by lemmas 6.2 and 6.3 we had that
$w(S) = \sum_{i\in S} \max\{\varphi^S(i) - \pi^S(i), 0\}$

Our aim now is to compare $\sum_{i\in S} m_i^{\pi,N}(w)$ and $w(S)$ using those expressions. Clearly if for all $k\in S$ it holds that $\max\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\} \geq \max\{\varphi^S(k) - \pi^S(k), 0\}$ then $\sum_{i\in S} m_i^{\pi,N}(w) \geq \sum_{i\in S} m_i^{\pi,S}(w) = w(S)$ and we are done, since then $m^{\pi,N}(w) \in C(w) \subseteq C(v)$. But this does not need to be true for all $k\in S$. Let $k\in S$ be such that

$$\max\{\varphi^S(k) - \pi^S(k), 0\} > \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}. \tag{30}$$

If this is the case we will focus on the study of the set of players $j\in N\setminus S$.

Let us introduce at this point some definitions of great help in the following. Let $k\in S$ satisfy (30). We define the player $j^*\in N\setminus S$ and $j^*\in PF_k^N(\pi,\varphi)$ such that

$$\left|\{j\in N\setminus S : j\in PF_k^N(\pi,\varphi) \text{ and } \varphi(j) < \varphi(j^*)\}\right| = \max\left\{\pi^{P_k(\pi)\cup S}(k) - \varphi^{P_k(\pi)\cup S}(k), 0\right\}$$

Moreover, let us define the set G_k as

$$G_k := \left\{ j \in N \backslash S : j \in PF_k^N(\pi,\varphi) \text{ and } \varphi(j) \geq \varphi(j^*) \right\}.$$

That j^* and G_k are well-defined will be shown in the proof of lemma 6.5. In lemma 6.5 we focus on the cardinality of the set G_k. In lemma **??** we focus on the difference $\pi^{P_j(\pi)\cup S}(j) - \varphi^{P_j(\pi)\cup S}(j)$ for players $j \in N\backslash S$.

Lemma 6.5 *Let $k \in S \subseteq N$ be such that*
$\max\left\{\varphi^S(k) - \pi^S(k), 0\right\} > \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}$.
Then it holds that
$|G_k| = \max\left\{\varphi^S(k) - \pi^S(k), 0\right\} - \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}$.

Proof. Let $k \in S$ satisfy (30).

We focus first on the cardinality of the set $\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}$. Applying (6) to k in $P_k(\pi) \cup S$ and k in S yields

$$\varphi^S(k) - \pi^S(k) = \left|FP_k^S(\pi,\varphi)\right| - \left|PF_k^S(\pi,\varphi)\right| \tag{31}$$

$$\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k) = \left|FP_k^{P_k(\pi)\cup S}(\pi,\varphi)\right| - \left|PF_k^{P_k(\pi)\cup S}(\pi,\varphi)\right|. \tag{32}$$

Subtracting (31) − (32) yields a useful expression for the cardinality of the set $\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}$:

$$\begin{aligned}
\varphi^S(k) - \pi^S(k) &= \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\
&+ \left|FP_k^S(\pi,\varphi)\right| - \left|PF_k^S(\pi,\varphi)\right| \\
&- \left|FP_k^{P_k(\pi)\cup S}(\pi,\varphi)\right| + \left|PF_k^{P_k(\pi)\cup S}(\pi,\varphi)\right| \\
&= \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\
&+ \left|PF_k^{P_k(\pi)\cup S}(\pi,\varphi)\right| - \left|PF_k^S(\pi,\varphi)\right| \\
&= \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\
&+ \left|PF_k^N(\pi,\varphi)\right| - \left|PF_k^S(\pi,\varphi)\right| \\
&= \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\
&+ \left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}\right|.
\end{aligned}$$

Here the second equality follows from $FP_k^S(\pi,\varphi) = FP_k^{P_k(\pi)\cup S}(\pi,\varphi)$ since the followers of k with respect to π in $P_k(\pi)\cup S$ are precisely the followers of k with respect to π in S. The third equality follows from the observation that $PF_k^{P_k(\pi)\cup S}(\pi,\varphi) = PF_k^N(\pi,\varphi)$ since the predecessors of k with respect to π in $P_k(\pi)\cup S$ are precisely the predecessors of k with respect to π in N. The last equality is trivial. Hence,

$$\left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}\right| = \varphi^S(k) - \pi^S(k) - \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right). \tag{33}$$

In order to determine the cardinality of G_k we distinguish between two cases:

Case I: $\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k) \geq 0$.

We first show that j^* is well-defined. By assumption of case I, clearly $\max\left\{\pi^{P_k(\pi)\cup S}(k) - \varphi^{P_k(\pi)\cup S}(k), 0\right\} = 0$. From (33) we have

$$\begin{aligned}
\left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}\right| &= \varphi^S(k) - \pi^S(k) \\
&- \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\
&> 0
\end{aligned}$$

where the inequality follows from the assumption of case I and the assumption of the lemma. So, j^* is well-defined. In particular, j^* is the first player in $(N\backslash S)\cap PF_k^N(\pi,\varphi)$ with respect to φ. This implies that

$$\begin{aligned} G_k &= \left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi) \text{ and } \varphi(j) \geq \varphi(j^*)\right\} \\ &= \left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} |G_k| &= \left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}\right| \\ &= \varphi^S(k) - \pi^S(k) - \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\ &= \max\left\{\varphi^S(k) - \pi^S(k), 0\right\} - \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}. \end{aligned}$$

Here the second equality follows from (33). The third equality follows from the assumption of case I and the assumption of the lemma. This completes the proof of case I.

Case II: $\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k) < 0$.

We first show that j^* is well-defined. Notice that from (33) it follows that

$$\begin{aligned} \left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi)\right\}\right| &= \varphi^S(k) - \pi^S(k) \\ &\quad - \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right) \\ &> 0. \end{aligned}$$

The inequality follows since the first term is positive by the assumption of the lemma and the second term is negative by the assumption of case II. Then the player j^* is well-defined and the set G_k, too.

Hence,

$$\begin{aligned} |G_k| &= \left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi) \text{ and } \varphi(j) \geq \varphi(j^*)\right\}\right| \\ &= \left|j \in N\backslash S : PF_k^N(\pi,\varphi)\right| \\ &\quad - \left|\left\{j \in N\backslash S : j \in PF_k^N(\pi,\varphi) \text{ and } \varphi(j) < \varphi(j^*)\right\}\right| \\ &= \left[\varphi^S(k) - \pi^S(k) - \left(\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k)\right)\right] \\ &\quad - \left[\pi^{P_k(\pi)\cup S}(k) - \varphi^{P_k(\pi)\cup S}(k)\right] \\ &= \varphi^S(k) - \pi^S(k) \\ &= \max\left\{\varphi^S(k) - \pi^S(k), 0\right\} - \max\left\{\varphi^{P_k(\pi)\cup S}(k) - \pi^{P_k(\pi)\cup S}(k), 0\right\}. \end{aligned}$$

Here first equality follows from the definition of G_k. The second and the fourth equalities are trivial. The third equality follows from (33) and the definition of j^*. The fifth equality follows from the assumption of case II and the assumption of the lemma. This completes the proof. □

Lemma 6.6 *For any player $j \in N\backslash S$ it holds that*

$$|\{k \in S : j \in G_k\}| \leq \max\left\{\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j), 0\right\}.$$

Proof. Let $j \in N\backslash S$ be such that $j \notin G_k$ for all $k \in S$ that satisfy (30). Clearly, $\max\left\{\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j), 0\right\} \geq 0$ and we are done.

Now, let $j \in N\backslash S$ be such that $|\{k \in S : j \in G_k\}| \geq 1$, i.e., j in G_k for some player $k \in S$ satisfying (30).

We define player $k^* \in S$ with $j \in G_{k^*}$ such that $\pi(k^*) > \pi(k)$ for all $k \in S$ with $j \in G_k$. In other words k^* is the last player of the set $\{k \in S : j \in G_k\}$ with respect to π.

In order to find a suitable expression for the difference $\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j)$ we apply (6) to k^* in $P_k(\pi)\cup S$ and j in $P_j(\pi)\cup S$,

$$\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) = \left|FP_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \tag{34}$$

$$\varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) = \left|FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right| - \left|PF_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right|. \tag{35}$$

Subtracting (34) − (35) yields,

$$\begin{aligned}
\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) \quad &= \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|FP_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| - \left|FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|PF_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&= \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|FP_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| - \left|FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&\geq \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|\{k \in S : j \in G_k\}\right|.
\end{aligned} \tag{36}$$

Here the second equality follows from the observation that $PF_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi) = PF_{k^*}^{N}(\pi,\varphi)$ since the predecessors of k^* with respect to π in $P_{k^*}(\pi)\cup S$ are precisely the predecessors of k^* with respect to π in N. To prove the inequality we will show that

$$\left|FP_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| - \left|FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)\right| \geq \left|\{k \in S : j \in G_k\}\right|. \tag{37}$$

The proof consists of two steps:

First step: Let $i \in FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)$, we will show that $i \in FP_j^{P_j(\pi)\cup S}(\pi,\varphi)$.

First, we see that $i \in P_j(\pi)\cup S$. Since i is a follower of k^* in $P_{k^*}(\pi)\cup S$ with respect to π, player i has to be in S and consequently in $P_j(\pi)\cup S$.

Second, we see that $\pi(j) < \pi(k^*) < \pi(i)$ and $\varphi(j) > \varphi(k^*) > \varphi(i)$. Here the first inequality in both expressions follows from $j \in G_{k^*}$ and hence, $j \in PF_{k^*}^{N}(\pi,\varphi)$. The second inequality in both expressions follows from $i \in FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)$. So, $i \in FP_j^{P_j(\pi)\cup S}(\pi,\varphi)$.

Second step: Let $k \in S$, $k \neq k^*$ be such that $j \in G_k$. We will show that $k \in FP_j^{P_j(\pi)\cup S}(\pi,\varphi)$ and $k \notin FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)$.

First, by assumption of $j \in G_k$, we have that $j \in PF_k^{N}(\pi,\varphi)$ and hence, $k \in FP_j^{N}(\pi,\varphi)$. Moreover, $k \in S$ and consequently, $k \in FP_j^{P_j(\pi)\cup S}(\pi,\varphi)$.

Second, by definition of k^* it holds that $\pi(k) < \pi(k^*)$ and consequently, $k \notin FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)$. Which follows easily since k cannot be a follower of k^* with respect to π. Notice that $k^* \in FP_j^{P_j(\pi)\cup S}(\pi,\varphi)$, but $k^* \notin FP_{k^*}^{P_{k^*}(\pi)\cup S}(\pi,\varphi)$.

From the two steps it follows that (37) holds. Hence, (36) holds. Hence,

$$\begin{aligned}
\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) \quad &\geq \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|\{k \in S : j \in G_k\}\right|.
\end{aligned} \tag{38}$$

We continue with studying expression (38). We distinguish between two cases:

Case I: $\varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \geq 0$.

In this case we can write:

$$\begin{aligned}
\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) &\geq \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|\{k \in S : j \in G_k\}\right| \\
&\geq \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|\{k \in S : j \in G_k\}\right| \\
&> \left|\{k \in S : j \in G_k\}\right|.
\end{aligned}$$

Here the first inequality follows from (38). The second inequality follows from the assumption of case I. To prove the third inequality we will show that $PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi) \subsetneqq PF_{k^*}^{N}(\pi,\varphi)$.

Let i be a player in $PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$ then we will show that $i \in PF_{k^*}^{N}(\pi,\varphi)$. First, clearly $i \in N$. Second, $\pi(i < \pi(j) < \pi(k^*)$ and $\varphi(i) > \varphi(j) > \varphi(k^*)$. Here the first inequality in both expressions follows from $i \in PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$. The second inequality in both expressions follows from $j \in G_{k^*}$ and hence, $j \in PF_{k^*}^{N}(\pi,\varphi)$. So, $i \in PF_{k^*}^{N}(\pi,\varphi)$. Notice that $j \in PF_{k^*}^{N}(\pi,\varphi)$, but $j \notin PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$.

Hence we can write,

$$\begin{aligned}
\max\left\{\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j), 0\right\} &\geq \varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) \\
&> \left|\{k \in S : j \in G_k\}\right|.
\end{aligned}$$

And this completes the proof of case I.

Case II $\varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) < 0$.

In this case we can write:

$$\begin{aligned}
\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) &\geq \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&+ \left|\{k \in S : j \in G_k\}\right| \\
&> \varphi^{P_{k^*}(\pi)\cup S}(k^*) - \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&- \varphi^{P_{k^*}(\pi)\cup S}(k^*) + \pi^{P_{k^*}(\pi)\cup S}(k^*) \\
&+ \left|\{k \in S : j \in G_k\}\right| \\
&= \left|\{k \in S : j \in G_k\}\right|.
\end{aligned}$$

Here the first inequality follows from (38). The equality is trivial. We prove the second inequality by showing that

$$\left|PF_{k^*}^{N}(\pi,\varphi)\right| - \left|PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)\right| > \pi^{P_{k^*}(\pi)\cup S}(k^*) - \varphi^{P_{k^*}(\pi)\cup S}(k^*).$$

The proof consists of three steps:

First step: $PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi) \subseteq PF_{k^*}^{N}(\pi,\varphi)$.

Let $i \in PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$ we will show that $i \in PF_{k^*}^{N}(\pi,\varphi)$. Clearly $i \in N$. Moreover, $\pi(i) < \pi(j) < \pi(k^*)$ and $\varphi(i) > \varphi(j) > \varphi(k^*)$. Here the first inequality in both expressions follows from $i \in PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$. The second inequality in both expressions follows from $j \in G_{k^*}$ and hence, $j \in PF_{k^*}^{N}(\pi,\varphi)$. So, $i \in PF_{k}^{N}(\pi,\varphi)$.

Second step: Let $i \in N\backslash S$; $i \in PF_{k^*}^{N}(\pi,\varphi)$ be such that $\varphi(i) < \varphi(j^*) \leq \varphi(j)$, where the second inequality follows from the assumption of $j \in G_{k^*}$. It is easy to see that $i \notin PF_{j}^{P_j(\pi)\cup S}(\pi,\varphi)$. Which follows since by the assumption of $\varphi(i) < \varphi(j)$, i cannot be a follower of j with respect to φ. Moreover, clearly $j \notin PF_j^{P_j(\pi)\cup S}(\pi,\varphi)$, but by the assumption of $j \in G_{k^*}$, $j \in PF_{k^*}^N(\pi,\varphi)$.

Third step: Now we are going to count the number of player of the set

$$\begin{aligned}
&\left|PF_{k^*}^N(\pi,\varphi)\right| - \left|PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&= \left|PF_{k^*}^N(\pi,\varphi)\backslash PF_j^{P_j(\pi)\cup S}(\pi,\varphi)\right| \\
&\geq |\{j\}| + \left|\{i \in N\backslash S : i \in PF_{k^*}^N(\pi,\varphi) \text{ and } \varphi(i) < \varphi(j)\}\right| \\
&> \left|\{i \in N\backslash S : i \in PF_{k^*}^N(\pi,\varphi) \text{ and } \varphi(i) < \varphi(j)\}\right| \\
&\geq \left|\{i \in N\backslash S : i \in PF_{k^*}^N(\pi,\varphi) \text{ and } \varphi(i) < \varphi(j^*)\}\right| \\
&= \pi^{P_{k^*}(\pi)\cup S}(k^*) - \varphi^{P_{k^*}(\pi)\cup S}(k^*)
\end{aligned}$$

Here the first equality follows from step one. The first inequality follows from step two. The second inequality is trivial. The third inequality follows from $j \in G_{k^*}$, and hence $\varphi(j^*) \leq \varphi(j)$. The last equality follows from the definition of j^* and the assumption of case II.

Hence we can write,

$$\begin{aligned}
\max\left\{\varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j), 0\right\} &\geq \varphi^{P_j(\pi)\cup S}(j) - \pi^{P_j(\pi)\cup S}(j) \\
&> \left|\{k \in S : j \in G_k\}\right|.
\end{aligned}$$

And this completes the proof of case II. □

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